Approximation of data using cubic Bézier curve least square fitting

Author: M Khan

Email: mak2000sw@yahoo.com
URL: `http://www.geocities.com/mak2000sw`

1 Bézier Curve

Bézier curve is a parametric curve. A Bézier curve of degree m can be generalized as follows:

$$q(t_i) = \sum_{k=0}^{m} \binom{m}{k} P_k (1-t_i)^{m-k} t_i^k, \quad 0 \le t_i \le 1, \tag{1}$$

where $q(t_i)$ is an interpolated point at parameter value t_i, m is degree of Bézier curve and P_k is k^{th} control point. To generate n points (n is count of interpolating points) between first and last control points inclusive, the parameter t_i is uniformly divided into $n-1$ intervals between 0 and 1 inclusive. Equations of cubic Bézier curves can be derived from Eq. (1) as follows:

$$q(t_i) = (1-t_i)^3 P_0 + 3t_i (1-t_i)^2 P_1 + 3t_i^2 (1-t_i) P_2 + t_i^3 P_3. \tag{2}$$

Bézier curve passes through its first and last control points i.e., P_0 and P_3. The *middle control points*, i.e. P_1 and P_2 determine the shape of curve.

2 Least Square Bézier Fitting

For data to be fit by cubic Bézier the first and last control points of Bézier curve are first and last point of the input data segment. The input data can be divided into many segments or just one segment by specifying the initial set of break points. But the the *middle control points*, i.e., P_1 and P_2 for cubic Bézier must be determined. We used *least square method* to find the *middle control points*. Least square method gives the *best* values of *middle control points* that minimize the squared distance between original and fitted data and is well suited for approximating data. If there are n data points and p_i and $q(t_i)$ are values of original and approximated points respectively then we can write the least square equation as follows:

$$S = \sum_{i=1}^{n} [p_i - q(t_i)]^2. \tag{3}$$

Eq. (3) can be written as follows:

$$S=\sum_{i=1}^{n}[p_i-(1-t_i)^3P_0+3t_i(1-t_i)^2P_1+3t_i^2(1-t_i)P_2+t_i^3P_3]^2. \tag{4}$$

P_1 and P_2 can be determined by:

$$\frac{\partial S}{\partial P_1}=0, \tag{5}$$

$$\frac{\partial S}{\partial P_2}=0. \tag{6}$$

Solving Eq. (5) for P_1 and Eq. (6) for P_2 gives:

$$P_1=(A_2C_1-A_{12}C_2)/(A_1A_2-A_{12}A_{12}), \tag{7}$$

$$P_2=(A_1C_2-A_{12}C_1)/(A_1A_2-A_{12}A_{12}), \tag{8}$$

where

$$A_1=9\sum_{i=1}^{n}t_i^2(1-t_i)^4, \tag{9}$$

$$A_2=9\sum_{i=1}^{n}t_i^4(1-t_i)^2, \tag{10}$$

$$A_{12}=9\sum_{i=1}^{n}t_i^3(1-t_i)^3, \tag{11}$$

$$C_1=\sum_{i=1}^{n}3t_i(1-t_i)^2[p_i-(1-t_i)^3P_0-t_i^3P_3], \tag{12}$$

$$C_2=\sum_{i=1}^{n}3t_i^2(1-t_i)[p_i-(1-t_i)^3P_0-t_i^3P_3]. \tag{13}$$

After determining the *control points*, Bézier curves can be fitted to large number of original data points with very few control points using Bézier interpolation.

3 Fitting Strategy

Suppose we have set of points (original data) $O=\{p_1,p_2,\ldots,p_n\}$ and we want to approximate it using cubic Bézier. As a input we specify the value of limit of error (maximum allowed square distance between original and fitted data) and provide initial set of breakpoints. At least two breakpoints are required i.e., the first point and the last point of original data. Input data is divided into segments based on initial set of breakpoints. A segment is set of all points between two consecutive breakpoints. We have to fit each segment using cubic Bézier curve(s). Now the fitting process begins. We generate n points (approximated data) $Q=\{q_1,q_2,\ldots,q_n\}$ using cubic Bézier interpolation such that cubic Bézier curve(s) passes through breakpoints. Then we measure the error between original and approximated (fitted) data.

When approximated data is not close enough to original data i.e. limit of error bound is violated then an existing segment is split (break) into two segments at a point called new breakpoint. After splitting number of segments are increased by one (splitted segment is replaced by two new segments). Number of breakpoints are also increased by one (new breakpoint is added in the set of existing breakpoints). The point where the error is maximum between original and approximated data is selected as new breakpoint and this point is added in the set of breakpoints.

After splitting repeat the same fitting procedure using updated set of segments and breakpoints until error is less than or equal to limit of error. We call this technique of fitting *break- and-fit* strategy.