I: Discrete Fourier transform and its Geometry

In this notes, the block structure of the matrix of the N-point discrete Fourier transform (DFT) in the real space R^{2N} is considered. All blocks 2×2 of this matrix correspond to the Givens transformations, or elementary rotations describing the multiplications by twiddle coefficients. Our goal is to illustrate the definition of the DFT as a beautiful system rotating the data on the plane around circles.

I. DFT in the real space

The N-point discrete Fourier transform of the signal is defined as the decomposition of the signal by N roots of the unit

$$W^k = W_N^k = e^{-i\frac{2\pi}{N}k} = c_k - is_k = \cos(\frac{2\pi}{N}k) - i\sin(\frac{2\pi}{N}k), \quad k = 0 : (N-1),$$

The roots are located on the unit circle, $(W^k)^N = 1$. The DFT of the vector-signal $\mathbf{f} = (f_0, f_1, f_2, ..., f_{N-1})'$, which is calculated by

$$F_p = R_p + iI_p = \sum_{n=0}^{N-1} W^{np} f_n, \quad p = 0 : (N-1),$$

has the following matrix in the complex space C^N:

$$[\mathcal{F}_N] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & & W^1 \end{bmatrix}.$$

Consider in matrix form the multiplication of the complex number $\mathbf{x} = x_1 + ix_2$, which is considered as the column-vector $(x_1, x_2)'$ or row-vector (x_1, x_2), by the twiddle coefficients W^k,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to W^k\mathbf{x} = \begin{pmatrix} c_k x_1 + s_k x_2 \\ c_k x_2 - s_k x_1 \end{pmatrix}, \quad k = 0 : (N-1),$$

where $W^k = (c_k, -s_k) = \cos\varphi_k - i\sin\varphi_k$, and the angles $\varphi_k = 2\pi k/N$. In matrix form, this multiplication can be written as

$$T^k\mathbf{x} = \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The matrix of rotation by the angle $\varphi_k = k\varphi_1$ is denoted by T^k. In the $k = 0$ case, $\varphi_0 = 0$, $T = I$, and there is no rotation of the signal-data.

Thus we transfer the complex plane into the 2-D real space, $C \to R^2$, and consider each operation of multiplication by the twiddle coefficient as the elementary rotation, or the Givens transformation, $W^k \to T^k$, $k = 0 : (N-1)$. We next consider the inclusion of the complex space C^N into the real space R^{2N} by

$$\mathbf{f} = (f_0, f_1, ..., f_{N-1})' \to \bar{\mathbf{f}} = (r_0, i_0, r_1, i_1, ..., r_{N-1}, i_{N-1})',$$

where we denote $r_k = \mathrm{Re} f_k$ and $i_k = \mathrm{Im} f_k$ for $k = 0 : (N-1)$. The vector $\bar{\mathbf{f}}$ is composed from the original vector, or signal $\mathbf{f}$, and its vector-component is denoted by $\bar{\mathbf{f}}_k = (\bar{f}_{2k}, \bar{f}_{2k+1})' = (r_k, i_k)'$. The N-point DFT of $\mathbf{f}$ is represented in the real space R^{2N} as the $2N$-point transform

$$\bar{\mathbf{F}}_p = \begin{bmatrix} R_p \\ I_p \end{bmatrix} = \sum_{k=0}^{N-1} T^{kp} \bar{\mathbf{f}}_k = \sum_{k=0}^{N-1} T^{kp} \begin{bmatrix} r_k \\ i_k \end{bmatrix} = \sum_{k=0}^{N-1} \begin{pmatrix} \cos(k\varphi_p) & \sin(k\varphi_p) \\ -\sin(k\varphi_p) & \cos(k\varphi_p) \end{pmatrix} \begin{bmatrix} r_k \\ i_k \end{bmatrix}, \tag{1}$$

where $p = 0 : (N-1)$.

In matrix form, the DFT in the space R^{2N} is described by the following matrix $2N \times 2N$:

$$X = \begin{bmatrix} I & I & I & I & I & I \\ I & T^1 & T^2 & T^3 & \cdots & T^{N-1} \\ I & T^2 & T^4 & T^6 & \cdots & T^{N-2} \\ I & \cdots & \cdots & \cdots & \cdots & \cdots \\ I & T^{N-1} & T^{N-2} & T^{N-3} & & T^1 \end{bmatrix}. \tag{2}$$

The (n,p)-th blocks 2×2 of this matrix is T^{np}, where $n, p = 0 : (N-1)$. The matrix $I = I_2$ is the identity matrix 2×2. Note the rotation matrices T^k compose the one-parametric group with period N. In other words

$$T^{k_1+k_2} = T^{k_1} T^{k_2}, \quad (T^0 = T^N = I),$$

for any $k_1, k_2 = 0 : (N-1)$.

According to (1), when calculating the component of the DFT at the frequency-point p, the vector-components, or points (r_k, i_k) are rotated by the corresponding angles $k\varphi_p$. Thus the first point (r_0, i_0) stays on its place, the second point (r_1, i_1) is rotated around the circle of radius $\sqrt{r_1^2 + i_1^2}$ by angle φ_p. The next point (r_2, i_2) is rotated around the circle of radius $\sqrt{r_2^2 + i_2^2}$ by twice larger angle $2\varphi_p$, or twice faster, than the point (r_1, i_1), and so on. Then the coordinates of all rotated points are added and one point is defined, which represents the DFT at frequency-point p. For point $p = 0$, there is no rotation; the sums of coordinates of the original points define the component F_0.

As example, consider the vector-signal of length 10 with the following real and imaginary parts:

$$\{r_k;\, k = 0:9\} = \{\ 5, \quad -3, \quad 2, \quad -5, \quad -1, \quad 3, \quad -4, \quad -2, \quad 7, \quad -6\}$$
$$\{i_k;\, k = 0:9\} = \{-2, \quad 1, \quad 4, \quad -3, \quad 7, \quad 2, \quad -5, \quad 5, \quad -1, \quad 3\}$$

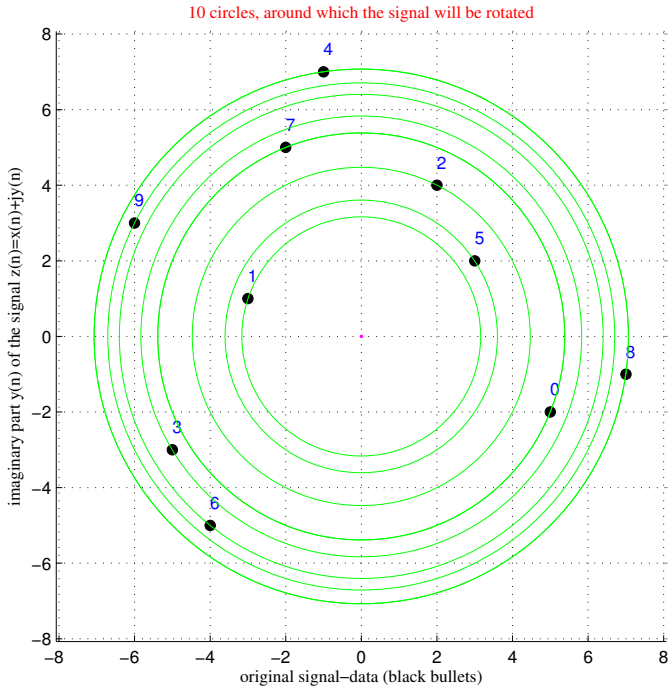

Fig. 1. Original signal of length ten with the circles.

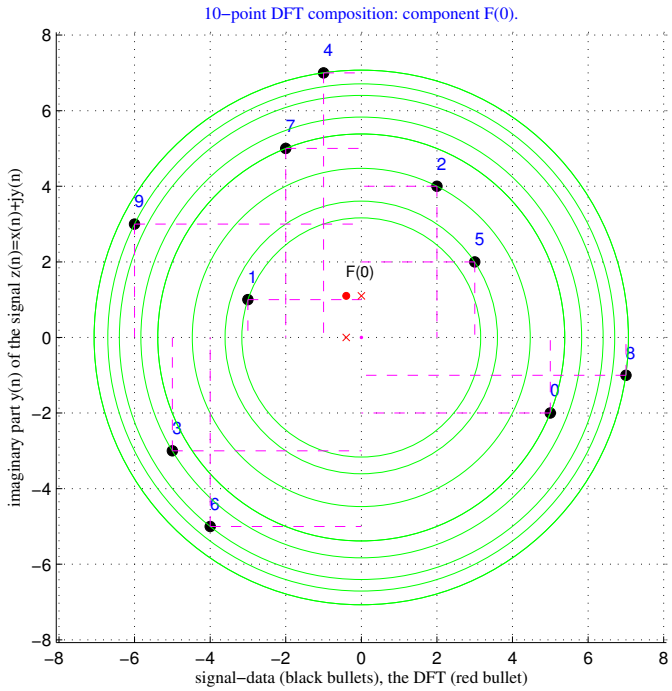

Fig. 2. Calculation of the first component of the DFT, F_0.

This signals is plotted in form of ten points on the plane in Figure 1. The points are numbered and ten circles on which they lie are shown in this figure.

Figure 2 shows projections of all points of the signal on the X-axis and Y-axis (shown by dash lines) and their sums (by x), and the point $F_0 = (-0.4, 1.1)$ (by the red bullet).

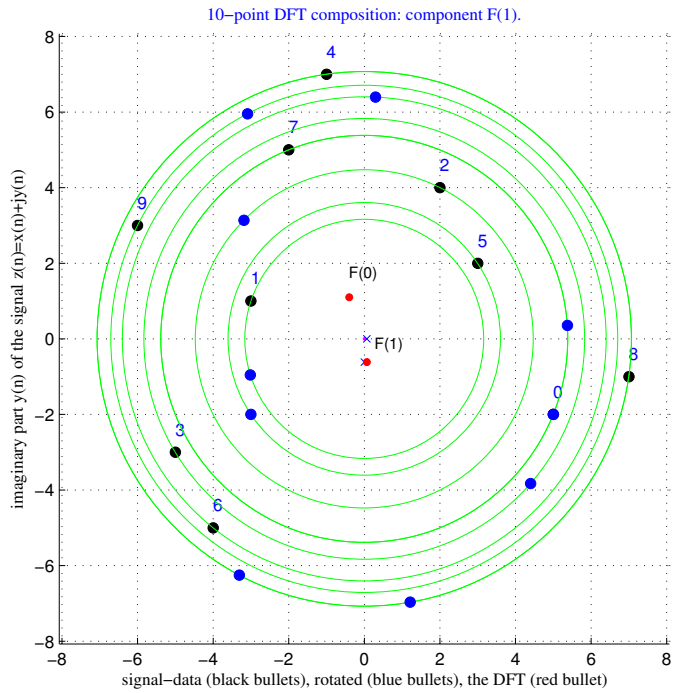

Fig. 3. Calculation of the second component of the DFT, F_1.

On the next stage of the DFT calculation, points of the signals are rotated as shown in Figure 3, and the sums of projections of the rotated points on the X-axis and Y-axis define the point $F_1 = (0.0684, -0.6155)$.

Similarly, these ten points are rotated eight more times, to arrive to ten points with the following coordinates:

$$\{R_p\} = \{-0.4, \quad 0.0684, \quad 0.7729, \quad -2.0982, \quad 4.0537, \quad 2.2, \quad -0.0648, \quad 0.1566, \quad -0.3618, \quad 0.6733\}$$
$$\{I_p\} = \{\ \ 1.1, \quad -0.6155, \quad -0.3367, \quad 0.4982, \quad -0.1407, \quad -0.5, \quad -0.5211, \quad -1.5836, \quad -0.1015, \quad 0.2009\}$$

where $p = 0 : 9$.

These ten points $F_p = (R_p, I_p)$ are show in Figure 4, as well as in Figure 5 in another view.

A. Inverse DFT

The inverse discrete Fourier transform (IDFT) is defined as

$$f_n = \frac{1}{N} \sum_{p=0}^{N-1} W^{-np} F_p, \quad n = 0 : (N-1). \tag{3}$$

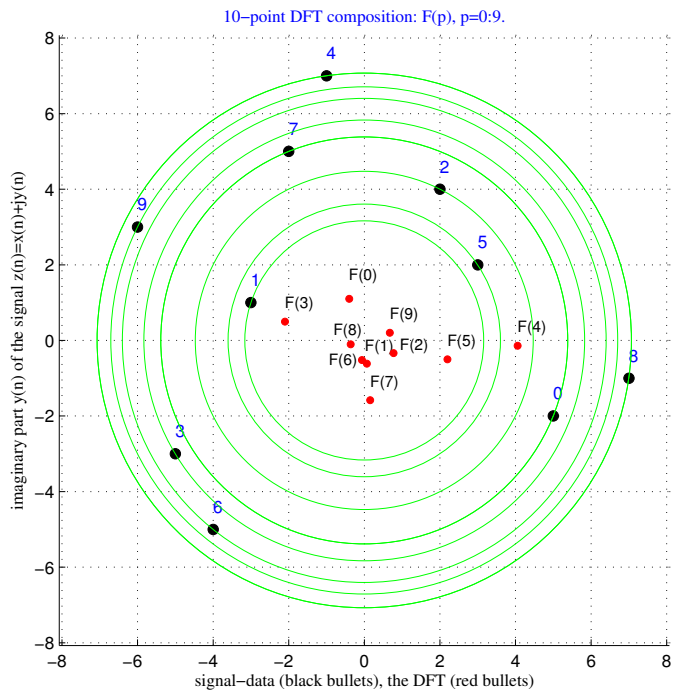

Fig. 4. Result of ten rotations of the signal-data.

In the real space R^{2N}, the N-point IDFT of $\{F_p;\, p = 0 : (N-1)\}$ is represented

$$\bar{\mathbf{f}}_n = \begin{bmatrix} r_n \\ i_n \end{bmatrix} = \frac{1}{N} \sum_{p=0}^{N-1} T^{-np} \begin{bmatrix} R_p \\ I_p \end{bmatrix} = \frac{1}{N} \sum_{p=0}^{N-1} \begin{pmatrix} \cos(n\varphi_p) & -\sin(n\varphi_p) \\ \sin(n\varphi_p) & \cos(n\varphi_p) \end{pmatrix} \begin{bmatrix} R_p \\ I_p \end{bmatrix}, \tag{4}$$

where $n = 0 : (N-1)$.

The illustration of the IDFT is similar to the DFT, with the following difference:

1. The IDFT is defined by rotation clock-wise, i.e. in the direction opposite to the rotation in the DFT.

2. After rotating of the points, the sums of the coordinates of rotated points are normalized by the factor of N.

Thus, if we start with the ten points $\{F_p;\, p = 0 : 9\}$ in the above considered example, as shown in Figure 6, the clock-wise rotations of these points around the circles on which they lie, will define the original signal, i.e. ten points with coordinates

$$\{r_k;\, k = 0 : 9\} = \{\ 5,\quad -3,\quad 2,\quad -5,\quad -1,\quad 3,\quad -4,\quad -2,\quad 7,\quad -6\}$$
$$\{i_k;\, k = 0 : 9\} = \{-2,\quad 1,\quad 4,\quad -3,\quad 7,\quad 2,\quad -5,\quad 5,\quad -1,\quad 3\}$$

The final rotation of ten points $\{F_p;\, p = 0 : 9\}$ is shown in Figure 7. One can see, that the result of the inverse rotation coincides with the points shown in Figure 1.

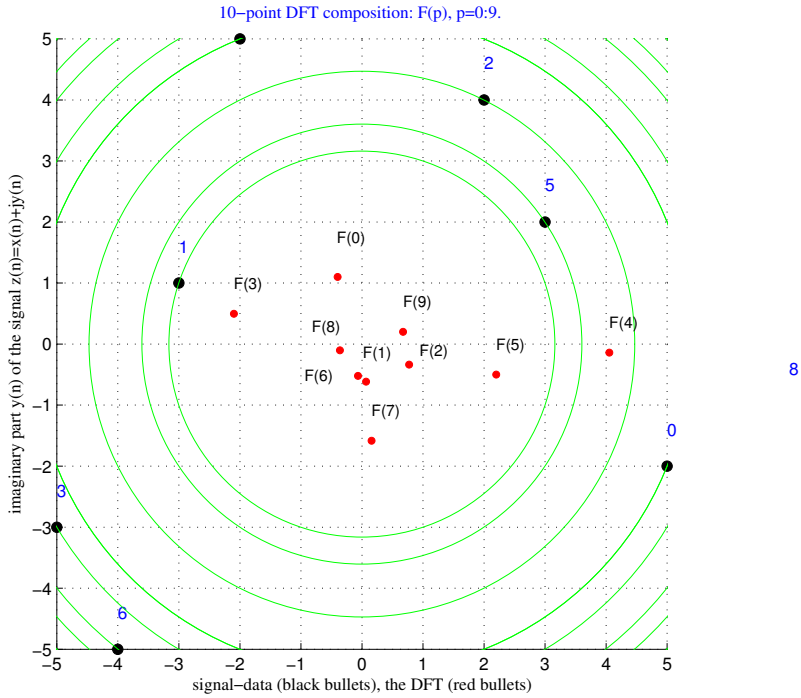

Fig. 5. Result of ten rotations of the signal-data.

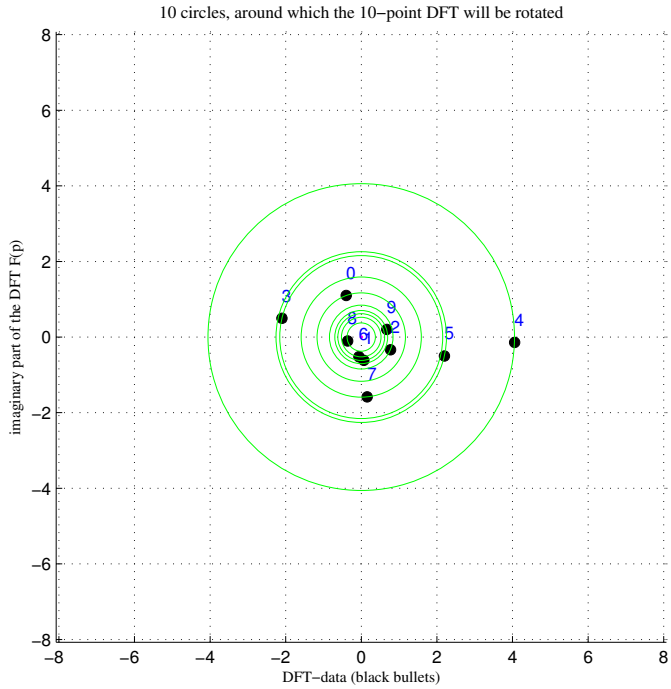

Fig. 6. Ten points of the inverse DFT on the circles.

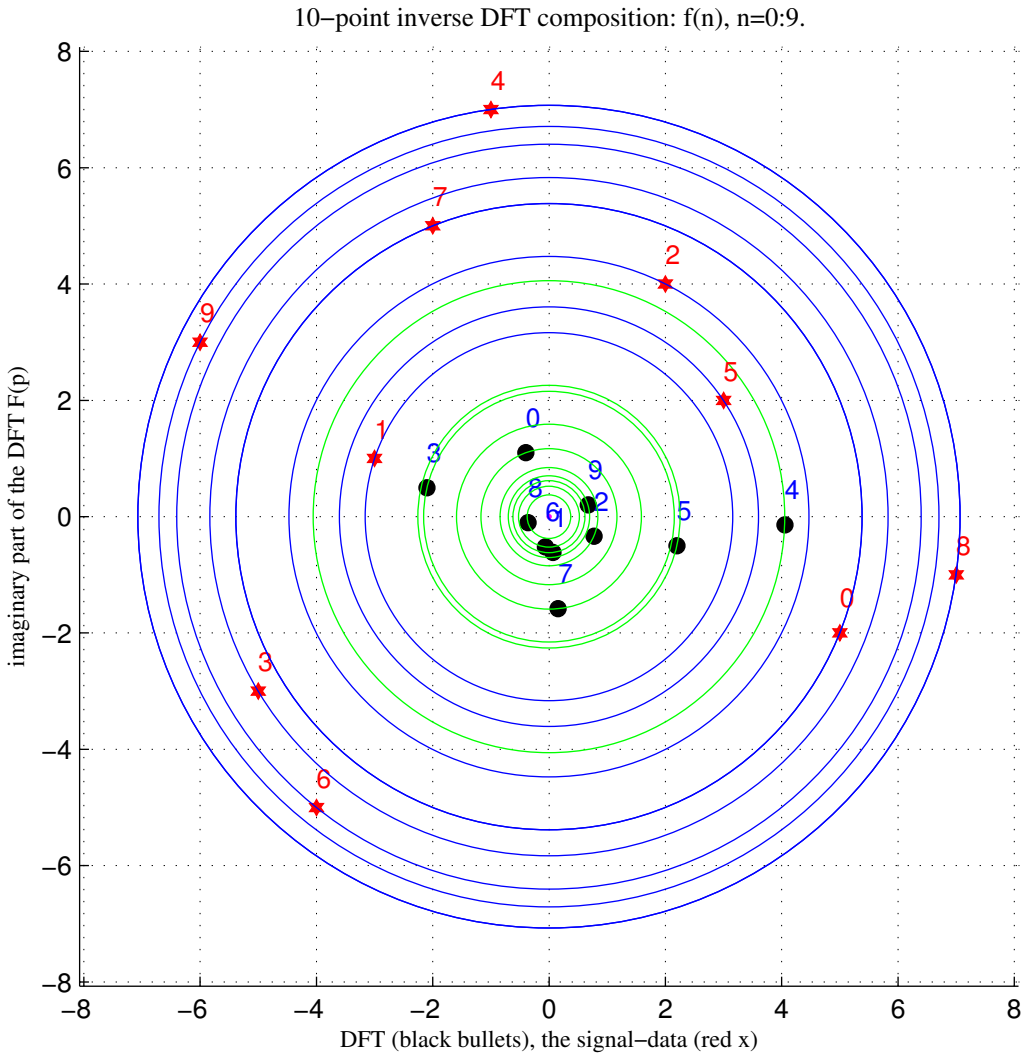

Fig. 7. The result of ten rotations of the inverse DFT-data.

A. DFT and Horoscope

Consider the following symmetric complex signal $z(n)$ of length $N = 201$,

$$z(n) = x(n) + iy(n) = n + i(n/5), \quad z(-n) = z(n), \qquad n = 0 : 200.$$

Figures 8-15 illustrates the geometry of a few components of the 201-point DFT. All these figures have been generated by the code "horoscope_1byDFTfiles.m" which is given in teh end of these notes.

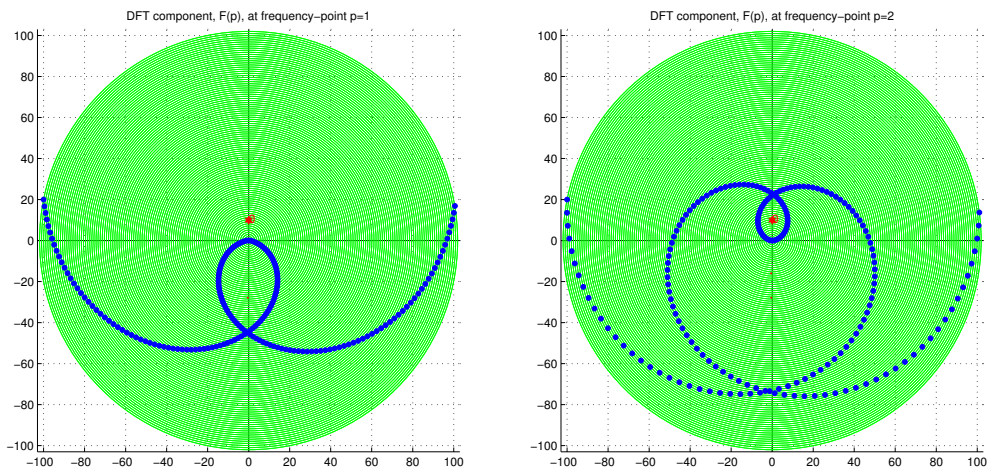

Fig. 8. DFT components F_1 and F_2.

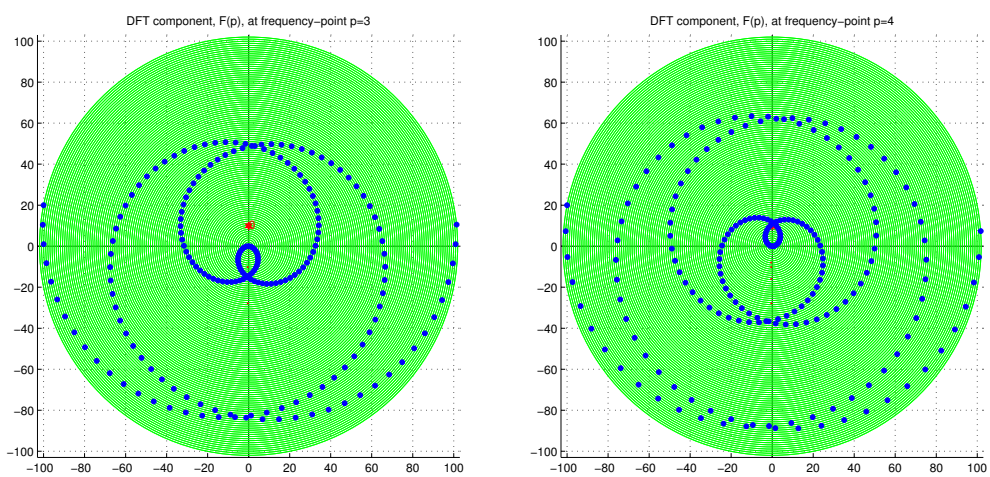

Fig. 9. DFT components F_3 and F_4.

The original signal is not real, i.e. $F_{N-p} \neq \bar{F}_p$, but we consider the geometry of such pairs for a few frequency-points, which are shown in Figures 16-19.

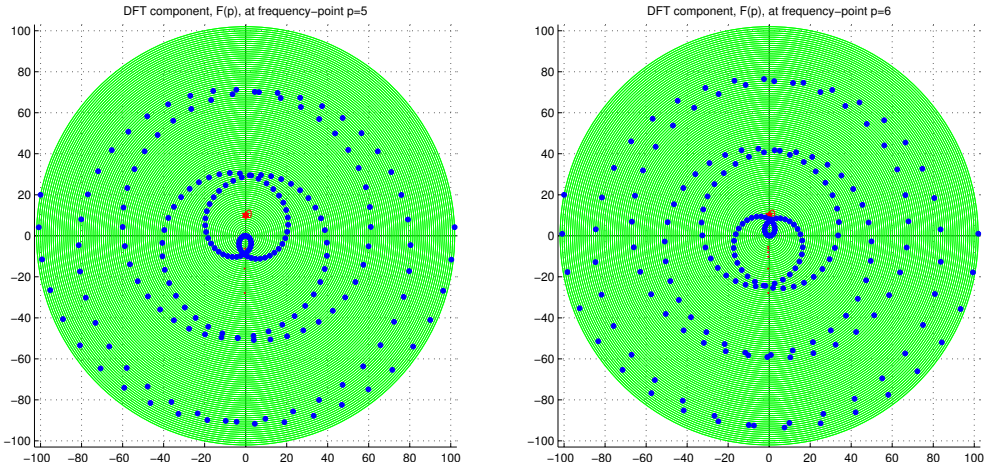

Fig. 10. DFT components F_5 and F_6.

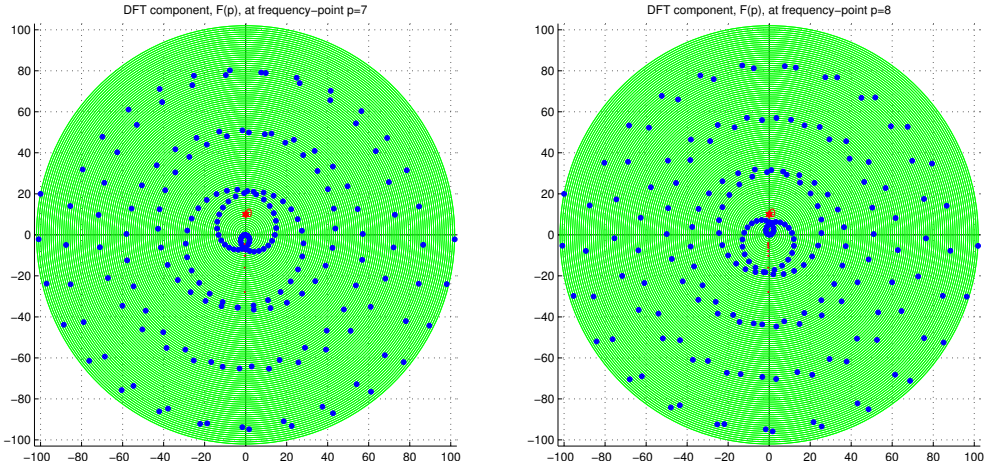

Fig. 11. DFT components F_7 and F_8.

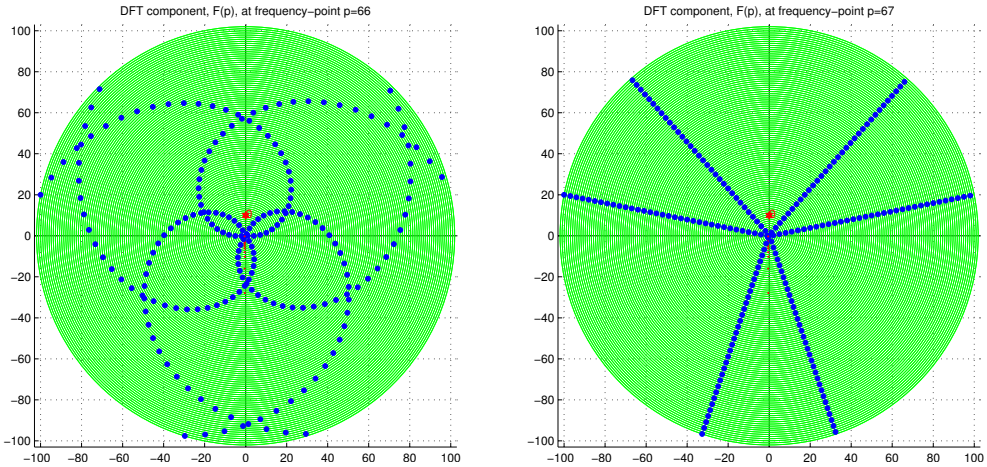

Fig. 12. DFT components F_{66} and F_{67}.

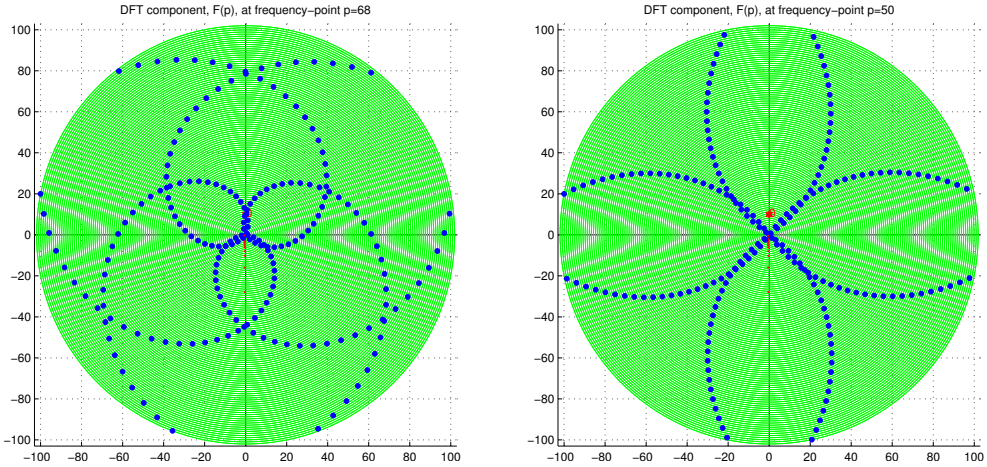

Fig. 13. DFT components F_{68} and F_{50}.

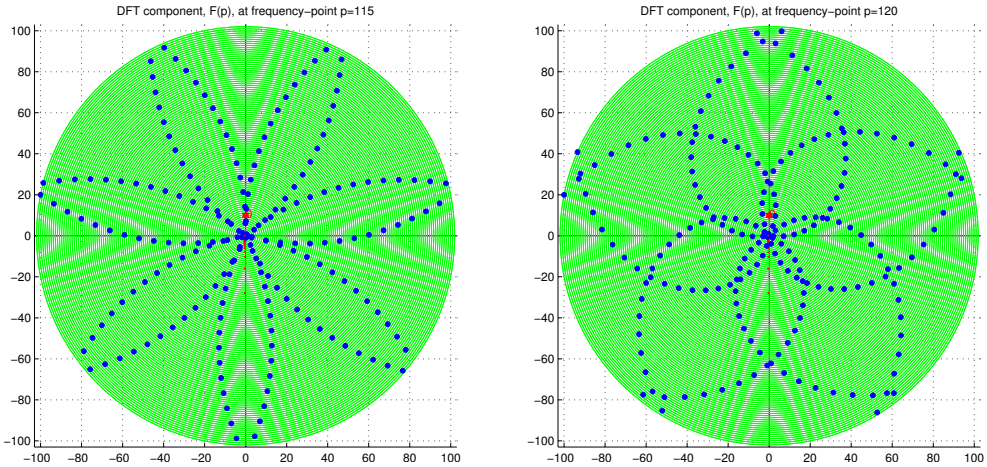

Fig. 14. DFT components F_{115} and F_{120}.

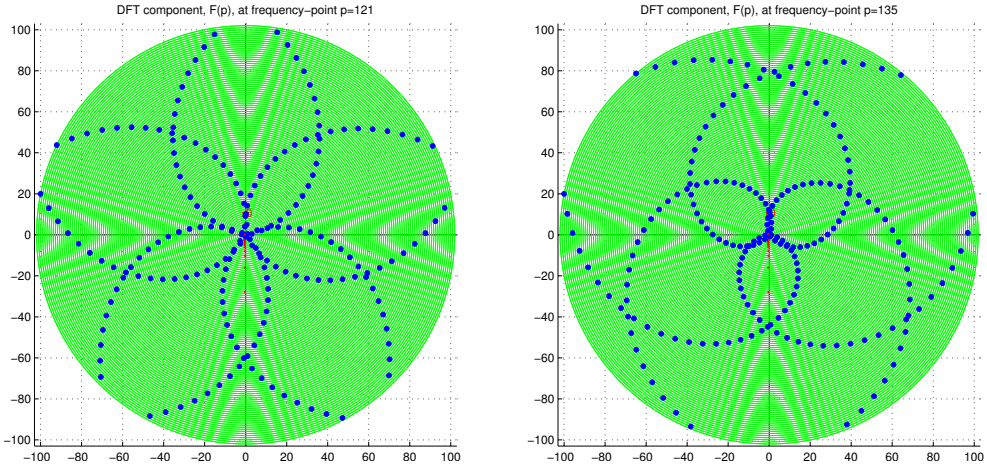

Fig. 15. DFT components F_{121} and F_{135}.

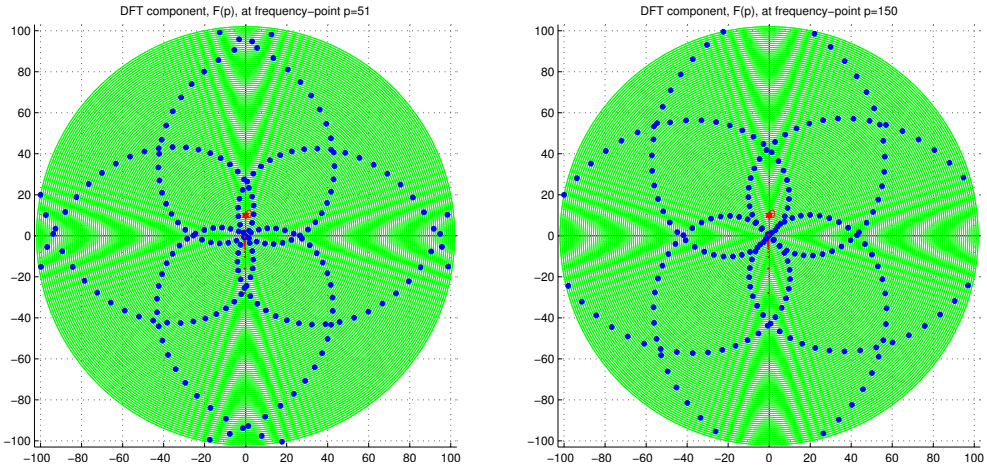

Fig. 16. DFT components F_{51} and F_{150}.

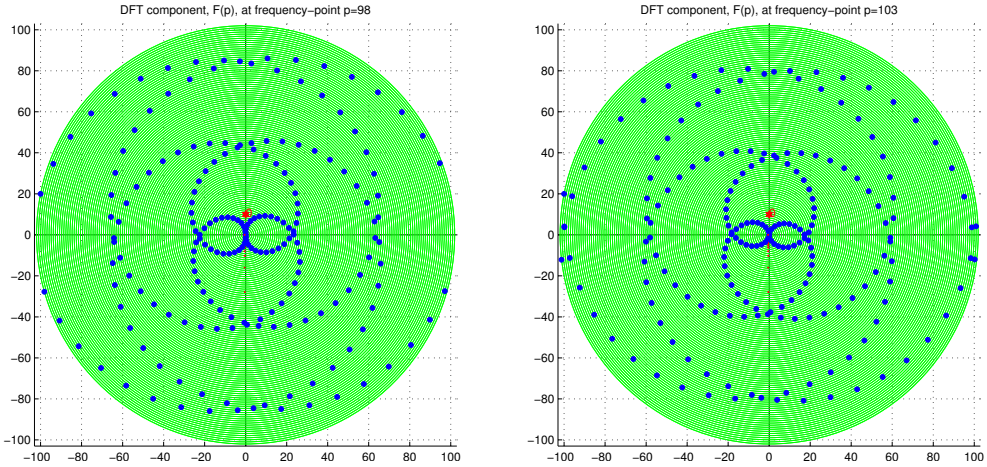

Fig. 17. DFT components F_{98} and F_{103}.

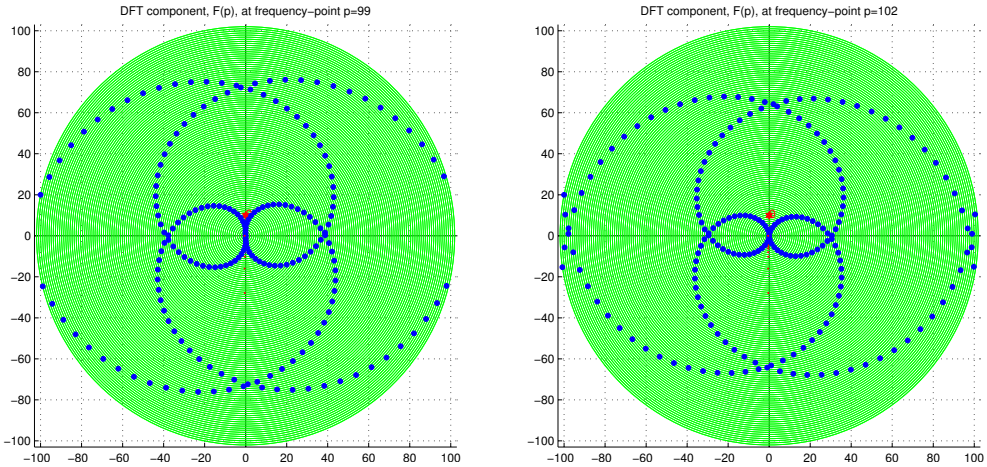

Fig. 18. DFT components F_{99} and F_{102}.

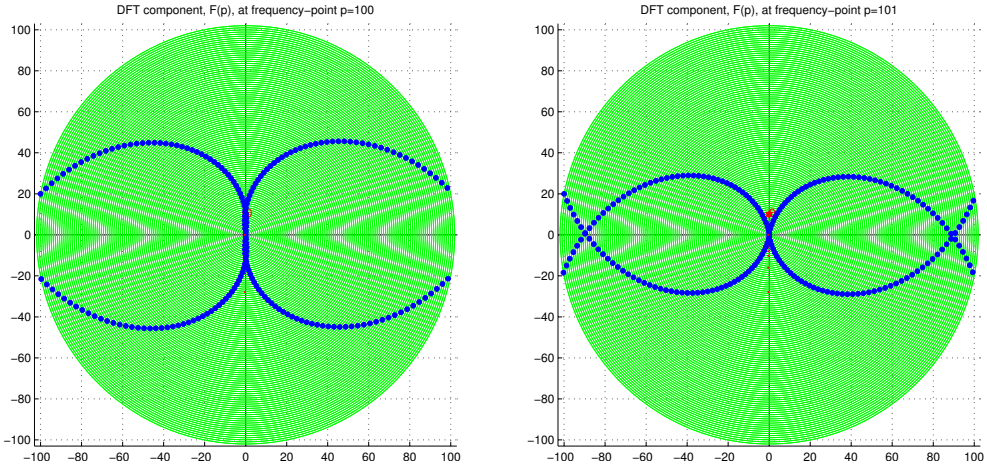

Fig. 19. DFT components F_{100} and F_{101}.