Stress Recovery

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1. Introduction

The main aim of any Finite Element Analysis is to get the stress values/stress distribution along the structure. In Finite Element Analysis, the main step is to solve the system of equations to get the nodal displacements. Once displacements are obtained, stresses are calculated using these displacements. Thus calculating the stresses is called post-processing step of the finite element method. The displacements are obtained directly from solving the equilibrium equation and are called as primary variables. As the stress values are obtained from these displacements, stress becomes derived quantities. As the finite element method is an approximate technique, the displacements have some error compared to exact values and thus the stress values have more error as they are derived from the displacements.

Therefore methods are developed to get the accurate stresses from the displacement values. The method should be computationally effective and should give better results. The procedure of extracting stresses from the displacements is called as *stress recovery* in finite element method.

2. Stress Calculation

The structural degrees of freedom {u} are calculated by solving the structural equations

$$[K]\{u\} = \{F\} \tag{2.1}$$

The basic way of calculating the stress field/ gradient field is to calculate it in element-by-element fashion. In elastic materials the stress field {σ} is related to the strain field {ε} by the elastic constitutive relation

$$\{\sigma\} = [D] \{\varepsilon\} \tag{2.2}$$

Where [D] is the constitutive material matrix depends on the material. The strains at any point in the element are related to the displacements as

$$\{\varepsilon\} = [B]\{u\}^e \tag{2.3}$$

Where [B] is the strain-displacement matrix containing derivatives of shape functions w.r.t x and y at the point where we need to find the stresses. Thus it is clear that to find the stresses, we need to find the strains and then from these strains, stresses are calculated. To calculate strains and stresses we need to perform a loop over all the elements in the domain. In eq. (2.3) the suffix *e* denotes the element.

Once strains are evaluated the corresponding stresses are given by

$$\{\sigma\} = [D] \{\varepsilon\} = [D] [B]\{u\} \tag{2.4}$$

In the present example, the stress recovery is applied to a two-dimensional plane stress problem discretized using four node isoparametric elements.

The stresses evaluated using eq. (2.4), are the stresses at the Gaussian points. But in general we are interested in finding these stresses at the element nodal points and at the midpoint of the element. These are called element nodal point stresses. These stresses do not display interelement continuity i.e. the stresses computed at the same nodal point will be different. But for displaying the stress distribution across the domain, stresses needs to be continuous for the better display. Thus the discontinuous nodal point stresses are made continuous using some operations called *smoothing* operations.

Let us see how the element nodal point stresses are computed from the stresses at the Gauss points. Three methods are available to get the element nodal point stresses.
1. Direct Evaluation
2. Extrapolation Calculation
3. Patch Recovery

A brief explanation of the above method is as follows.

2. 1 Direct Evaluation

In this method, {σ} is evaluated directly at the element node locations by substituting the natural coordinates of the nodal points as arguments in the shape functions. These shape functions return element nodal displacements ux and uy. On substituting these nodal displacements in eq. (2.4) yields the stresses at the nodes. This method is a simple method and easy to implement.
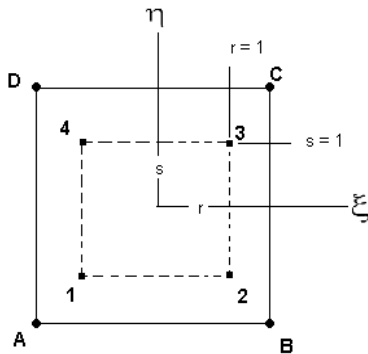
2.2 Extrapolation Calculation

Figure 1: Reference coordinate systems used in extra-polation of stresses from Gauss points. 2 by 2 Gauss point quadrature is used

Imagine that the stresses at the four Gauss points 1, 2, 3, 4 as shown in Fig. 1 have been calculated. The points are shown in counter clock wise direction. The Gauss points follow the ordering of the element node numbering. For the Gauss points 1, 2, 3 and 4 the respective element nodes are A, B, C and D respectively. Using the stresses at the Gauss points, the stresses at the element nodes are calculated by a bilinear extrapolation using the shape functions.

To understand it more clearly, let us associate a ξη frame for the element and rs frame for the Gauss element. (The region bounded by the four Gauss points is the Gauss element). The Gauss element is also a four node quadrilateral. The dimensionless coordinate's r and s of the Gauss element are respectively proportional to ξ and η. At a point say 3, we have r = s = 1 and $\xi = \eta = \frac{1}{\sqrt{3}}$. Therefore the proportionality factor is $\sqrt{3}$. And we have

$$r = \sqrt{3}\,\xi \text{ and } s = \sqrt{3}\eta \tag{2.2.1}$$

The natural coordinates and proportionality factors for the element and Gauss element are listed in the table 1. Stress at any point P in the element can be obtained by the usual shape unction formula, with shape functions N_i', which are evaluated at the coordinates of point P:

$$\sigma_p = \sum N_i' \sigma_i \quad \text{for} \quad i = 1,2,3,4 \tag{2.2.2}$$

$$= [\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4] \begin{bmatrix} N_1' \\ N_2' \\ N_3' \\ N_4' \end{bmatrix} \tag{2.2.3}$$

Where σ can be σ$_x$, σ$_y$ or τ$_{xy}$. The N_i' are bilinear shape functions of 4 noded isoparametric elements with ξ and η replaced by r and s;

$$N_1' = \frac{1}{4}(1-r)(1-s)$$

$$N_2' = \frac{1}{4}(1+r)(1-s)$$

$$N_3' = \frac{1}{4}(1+r)(1+s)$$

$$N_4' = \frac{1}{4}(1-r)(1+s) \tag{2.2.4}$$

For example, let us calculate stress σ$_x$ at the corner A from the σ$_x$ values at Gauss points 1, 2, 3, 4. We have to substitute r = s = -√3 and from eq. (2.2.2) we have

$$\sigma_{xA} = \left[1 + \frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad 1 - \frac{\sqrt{3}}{2} \quad -\frac{1}{2}\right] \begin{bmatrix} \sigma_{x1} \\ \sigma_{x2} \\ \sigma_{x3} \\ \sigma_{x4} \end{bmatrix} \tag{2.2.5}$$

Doing the above for all the four corners we get:

$$\begin{bmatrix} \sigma_{xA} \\ \sigma_{xB} \\ \sigma_{xC} \\ \sigma_{xD} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \\ 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \sigma_{x1} \\ \sigma_{x2} \\ \sigma_{x3} \\ \sigma_{x4} \end{bmatrix} \tag{2.2.6}$$

Check that in the above 4X4 matrix, sum of the coefficients in each row is one. The extrapolation method for the higher order elements also follows the same procedure.

Table 1: Natural coordinates of element and Gauss element

Element Nodes	ξ	η	r	s	Gauss nodes	ξ	η	r	s
A	-1	-1	$-\sqrt{3}$	$-\sqrt{3}$	1	$-\dfrac{1}{\sqrt{3}}$	$-\dfrac{1}{\sqrt{3}}$	-1	-1
B	+1	-1	$+\sqrt{3}$	$-\sqrt{3}$	2	$+\dfrac{1}{\sqrt{3}}$	$-\dfrac{1}{\sqrt{3}}$	+1	-1
C	+1	+1	$+\sqrt{3}$	$+\sqrt{3}$	3	$+\dfrac{1}{\sqrt{3}}$	$+\dfrac{1}{\sqrt{3}}$	+1	+1
D	-1	+1	$-\sqrt{3}$	$+\sqrt{3}$	4	$-\dfrac{1}{\sqrt{3}}$	$+\dfrac{1}{\sqrt{3}}$	-1	+1

2.3 Patch Recovery

In this recovery process, it is assumed that stress at nodal values σ^* belong to a polynomial expansion $\boldsymbol{\sigma_p^*}$ of the same complete order p as that present in the basis function N and which is valid over an element patch surrounding the particular assembly node considered. A small number of contiguous elements around a node is called a patch. A typical patch for four noded isoparametric elements is shown in Fig. 2. The polynomial expansion is written as
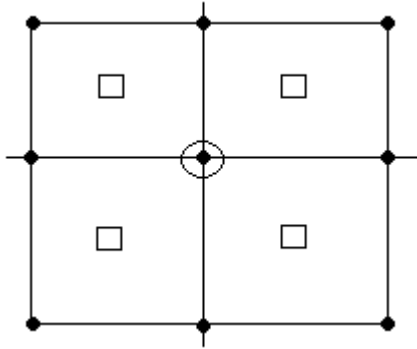

$$\sigma_p^* = [P]\{a\} \qquad (2.3.1)$$

Where [P] contains the terms of appropriate polynomial and {a} is a set of unknown parameters which are supposed to be determined. For a plane problem the possible choices of the polynomial matrix are

Bilinear: [P] = [1 x y xy]   &   {a} = {$a_1\ a_2\ a_3\ a_4$}

Quadratic: [P] = [1 x y x^2 xy y^2] & {a} = {$a_1\ a_2\ a_3\ a_4\ a_5\ a_6$}

$$(2.3.2)$$

Figure 2: Patch for four noded element

To determine {a}, a patch is selected as shown in Fig. 2, and assign a coordinate system xy. The origin of the is located arbitrarily such that it is close to the patch and numerical differences are lower. The element stress field is sampled at locations where it is more accurate. These locations are the Gauss points. Thus the set of the unknown parameters {a} are determined by ensuring a least square fit of the unknown stresses with the stresses at the Gauss points. To do this we minimize

$$F_P = \sum_{i=1}^{n}(\sigma_p^* - \sigma)_i^2 \qquad (2.3.3)$$

Where n is the number of sampling points in the patch. The minimization condition of eq. (2.3.3) gives

$$[A] = \sum_{i=1}^{n}[P]_i^T[P]_i$$

[A]{a} = {b}   where $\qquad (2.3.4)$

$$\{b\} = \sum_{i=1}^{n}[P]_i^T \sigma_i$$

The above one can be solved to get unknowns {a} as follows:

$$\{a\} = [A]^{-1}\{b\} \tag{2.3.5}$$

The same matrix [A] is used to find all stress components in a problem. Once {a} is known the smoothed stress/ element nodal stress at any point can be evaluated by substituting the x and y coordinates of the point into the eq. (2.3.1).

3. Nodal Averaging

Let a stress σ_e be computed at each node in each element, either directly at nodes or by extrapolation to nodes from Gauss points. At a node shared by n elements there are in general n different values of σ_e. The nodal average is given by

$$(\sigma_e)_{avg} = \frac{1}{n}\sum_{i=1}^{n}(\sigma_e)_i \tag{3.1}$$

Sometimes stress at each node in each element is summed without averaging it.

$$(\sigma_e)_{avg} = \sum_{i=1}^{n}(\sigma_e)_i \tag{3.2}$$

4. Von Mises Stress

Some stress quantities are invariant; i.e. they have the same numerical value irrespective of the coordinate system in which they are computed. One such quantity is the Von Mises stress or effective stress. The stress $\{\sigma\}$ in case of 2D plane stress problem have the stress components σ_x, σ_y and τ_{xy}. Theses stress components can be written in matrix form or Cauchy stress as

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{4.1}$$

The von Mises stress is given by

$$\sigma_{vim} = \frac{1}{\sqrt{2}}\left(\left(\sigma_x - \sigma_y\right)^2 + \sigma_y^2 + 6\tau_{xy}^2\right)^2 \tag{4.2}$$

The Von Mises stress can also be calculated from principal stresses as

$$\sigma_{vim} = \frac{1}{\sqrt{2}}\left((\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_2^2\right)^2 \tag{4.3}$$

Where the principal stress σ_1, σ_2 and σ_3 are given by

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2} \tag{4.4}$$

Maximum and minimum shear stresses are given byType equation here.

$$\tau_{max} = +\sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2} \qquad (4.5)$$

$$\tau_{min} = -\sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2} \qquad (4.6)$$