Bi-elliptic Transfer Between Coplanar Circular Orbits

This document describes a MATLAB script called `bielliptic.m` that can be used to determine the characteristics of a time-free, three impulse bi-elliptic transfer between two coplanar circular orbits. The impulsive ΔV assumption means that the velocity, but not the position, of the space vehicle is changed instantaneously. This script also creates graphic displays of the three-dimensional orbits and transfer trajectory, and the evolution of the primer vector and its derivative.

For the coplanar bi-elliptic transfer, all three velocity impulses are confined to the orbital planes of the initial and final orbits. The first impulse creates an elliptical transfer orbit with a perigee altitude equal to the altitude of the initial circular orbit and an apogee altitude well beyond the altitude of the final orbit. The second impulse creates a second transfer ellipse with an apogee radius equal to that of the first transfer ellipse and a perigee radius identical to the value of the final orbit radius.

The third impulsive maneuver circularizes the second transfer orbit at perigee of the transfer orbit. The first two impulses are *posigrade* which means that they are in the direction of orbital motion. The third impulse is retrograde since it must slow down the spacecraft for insertion into the final mission orbit.

For final to initial radius ratios greater than 15.58, the bi-elliptic transfer requires less total propulsive energy than the Hohmann transfer. It can also be shown that an <u>outer</u> (intermediate apogee altitude greater than the final orbit altitude) bi-elliptic transfer is more efficient than an inner transfer.

Interacting with the script

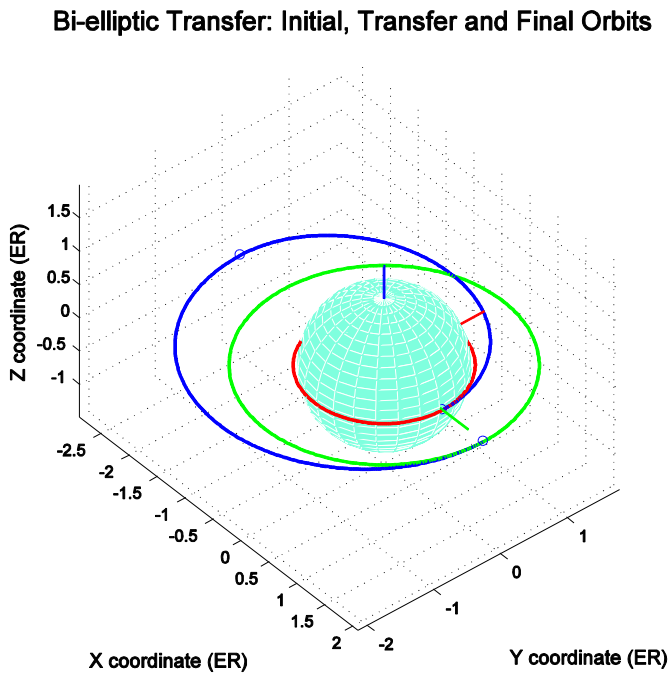
The following is a typical user interaction with this script. User inputs to the script are in bold font. Please note that the script will either accept a user-defined intermediate altitude or calculate the optimum value using Brent's root-finding algorithm.

```
Bi-elliptic Orbit Transfer Analysis


please input the initial altitude (kilometers)
? 300.0


please input the final altitude (kilometers)
? 5000.0


type of intermediate altitude computation

  <1> optimal

  <2> user-defined

 selection (1 or 2)
? 2

please input the bi-elliptic altitude (kilometers)
? 10000.0
```

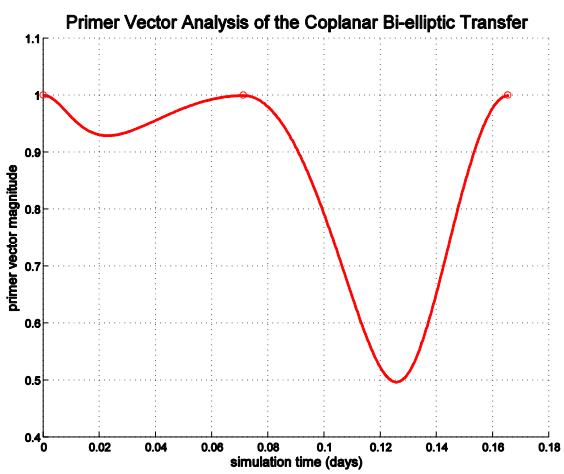
The following is the script output created for this example.

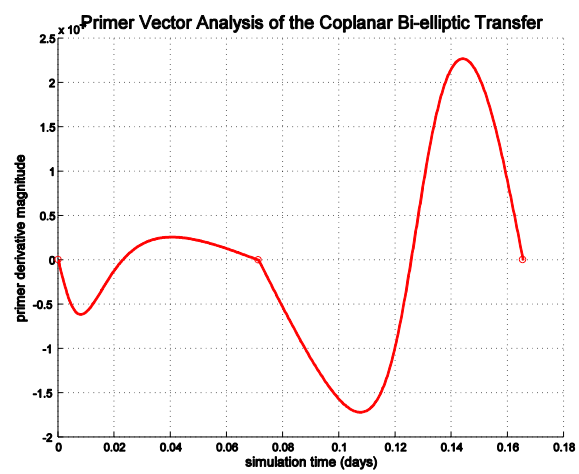
```
Bi-elliptic Orbit Transfer Analysis
-----------------------------------

initial orbit altitude                 300.0000 kilometers

initial orbit radius                  6678.1363 kilometers

initial orbit velocity                7725.7606 meters/second

first ellipse perigee altitude         300.0000 kilometers

first ellipse perigee radius          6678.1363 kilometers

first ellipse apogee altitude        10000.0000 kilometers

first ellipse apogee radius          16378.1363 kilometers

first ellipse perigee velocity        9208.6069 meters/second

first ellipse apogee velocity         3754.7820 meters/second

first ellipse eccentricity           0.42070981

second ellipse perigee altitude       5000.0000 kilometers

second ellipse perigee radius        11378.1363 kilometers

second ellipse apogee altitude       10000.0000 kilometers

second ellipse apogee radius         16378.1363 kilometers

second ellipse perigee velocity       6429.8373 meters/second

second ellipse apogee velocity        4466.9042 meters/second

second ellipse eccentricity          0.18013946

final orbit altitude                  5000.0000 kilometers

final orbit radius                   11378.1363 kilometers

final orbit velocity                  5918.7953 meters/second

first delta-v                         1482.8463 meters/second

second delta-v                         712.1221 meters/second

third delta-v                          511.0420 meters/second

total delta-v                         2706.0105 meters/second

first ellipse transfer time              1.7109 hours
                                         0.0713 days

second ellipse transfer time             2.2598 hours
                                         0.0942 days

total transfer time                      3.9707 hours
                                         0.1654 days
```

The `bielliptic` MATLAB script will also create graphic displays of the three-dimensional initial, final and transfer trajectory and the evolution of the primer vector and its derivative. The first graphic image is a three-dimensional display of the solution. In this image, the initial orbit is red, the final orbit is green, and the transfer trajectory is blue. The dimensions are Earth radii (ER) and the plot is labeled with an ECI coordinate system where green is the x-axis, red is the y-axis and blue is the z-axis. The interactive graphic features of MATLAB allow the user to rotate and zoom the display. These capabilities allow the user to interactively find the best viewpoint as well as verify basic three-dimensional geometry of the orbital transfer.



These next two plots illustrate the evolution of the primer vector and its derivative as a function of time, in days, since the first impulse. The location of each impulse is marked with a small red circle.





The `bielliptic` MATLAB script will also create color Postscript disk files of these graphic images. Each image includes a TIFF preview and is created with MATLAB source code similar to

```
print -depsc -tiff -r300 bielliptic1.eps
```

For comparison, here are the characteristics of this mission using a two impulse Hohmann transfer.

```
Hohmann Orbit Transfer Analysis
-------------------------------

initial orbit altitude              300.0000 kilometers

initial orbit radius               6678.1363 kilometers

initial orbit inclination             0.0000 degrees

initial orbit velocity             7725.7606 meters/second


final orbit altitude               5000.0000 kilometers

final orbit radius                11378.1363 kilometers

final orbit inclination               0.0000 degrees

final orbit velocity               5918.7953 meters/second


first inclination change              0.0000 degrees

second inclination change             0.0000 degrees

total inclination change              0.0000 degrees


first delta-v                       947.4074 meters/second

second delta-v                      828.2781 meters/second

total delta-v                      1775.6855 meters/second


transfer orbit semimajor axis      9028.1363 kilometers

transfer orbit eccentricity       0.26029736

transfer orbit inclination            0.0000 degrees

transfer orbit perigee velocity    8673.1680 meters/second

transfer orbit apogee velocity     5090.5171 meters/second

transfer orbit coast time          4268.5281 seconds
                                     71.1421 minutes
                                      1.1857 hours
```

Since the radius ratio $\left(r_f / r_i\right)$ for this example is less than 15.58, the Hohmann orbit transfer is more efficient, in terms of the total ΔV required, than the bi-elliptic transfer.

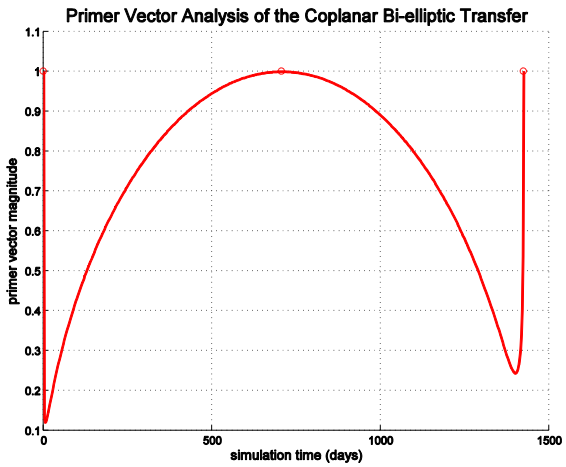
The following is the script output for this same example where we allow the software to compute the optimal intermediate altitude. Since the radius ratio is less than 15.58, the script finds a two impulse Hohmann transfer. For this situation, the apogee altitude of the intermediate transfer ellipses is equal to the altitude of the final circular orbit.

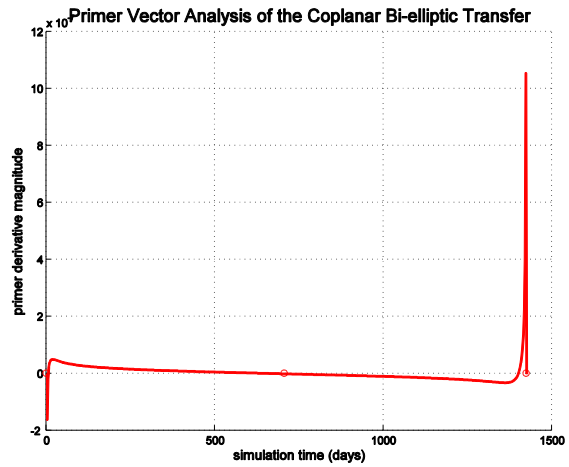
```
Bi-elliptic Orbit Transfer Analysis
-----------------------------------

initial orbit altitude                 300.0000 kilometers

initial orbit radius                  6678.1363 kilometers

initial orbit velocity                7725.7606 meters/second

first ellipse perigee altitude         300.0000 kilometers

first ellipse perigee radius          6678.1363 kilometers

first ellipse apogee altitude         5000.0003 kilometers

first ellipse apogee radius          11378.1366 kilometers

first ellipse perigee velocity        8673.1680 meters/second

first ellipse apogee velocity         5090.5170 meters/second

first ellipse eccentricity           0.26029737

second ellipse perigee altitude       5000.0000 kilometers

second ellipse perigee radius        11378.1363 kilometers

second ellipse apogee altitude        5000.0003 kilometers

second ellipse apogee radius         11378.1366 kilometers

second ellipse perigee velocity       5918.7953 meters/second

second ellipse apogee velocity        5918.7952 meters/second

second ellipse eccentricity          0.00000001

final orbit altitude                  5000.0000 kilometers

final orbit radius                   11378.1363 kilometers

final orbit velocity                  5918.7953 meters/second

first delta-v                          947.4074 meters/second

second delta-v                         828.2781 meters/second

third delta-v                            0.0000 meters/second

total delta-v                         1775.6856 meters/second

first ellipse transfer time              1.1857 hours
                                         0.0494 days

second ellipse transfer time             1.6776 hours
                                         0.0699 days

total transfer time                      2.8633 hours
                                         0.1193 days
```

Here's the script output and primer plots for an orbit transfer example where we allow the software to compute the optimal apogee altitude of the two intermediate transfer ellipses. For this example, the altitude of the initial circular orbit is 300 kilometers and the altitude of the final circular orbit is 100,000 kilometers $\left(r_f / r_i = 15.929\right)$.

```
Bi-elliptic Orbit Transfer Analysis
-----------------------------------

initial orbit altitude                 300.0000 kilometers

initial orbit radius                  6678.1363 kilometers

initial orbit velocity                7725.7606 meters/second

first ellipse perigee altitude         300.0000 kilometers

first ellipse perigee radius          6678.1363 kilometers

first ellipse apogee altitude     10631435.2731 kilometers

first ellipse apogee radius       10637813.4094 kilometers

first ellipse perigee velocity       10922.4475 meters/second

first ellipse apogee velocity            6.8568 meters/second

first ellipse eccentricity           0.99874524

second ellipse perigee altitude     100000.0000 kilometers

second ellipse perigee radius       106378.1363 kilometers

second ellipse apogee altitude    10631435.2731 kilometers

second ellipse apogee radius      10637813.4094 kilometers

second ellipse perigee velocity       2723.9367 meters/second

second ellipse apogee velocity          27.2394 meters/second

second ellipse eccentricity          0.98019802

final orbit altitude                100000.0000 kilometers

final orbit radius                  106378.1363 kilometers

final orbit velocity                  1935.7207 meters/second

first delta-v                         3196.6869 meters/second

second delta-v                          20.3825 meters/second

third delta-v                          788.2160 meters/second

total delta-v                         4005.2855 meters/second

first ellipse transfer time          16971.5253 hours
                                       707.1469 days
```

```
second ellipse transfer time          17210.5245 hours
                                        717.1052 days

total transfer time                   34182.0498 hours
                                       1424.2521 days
```





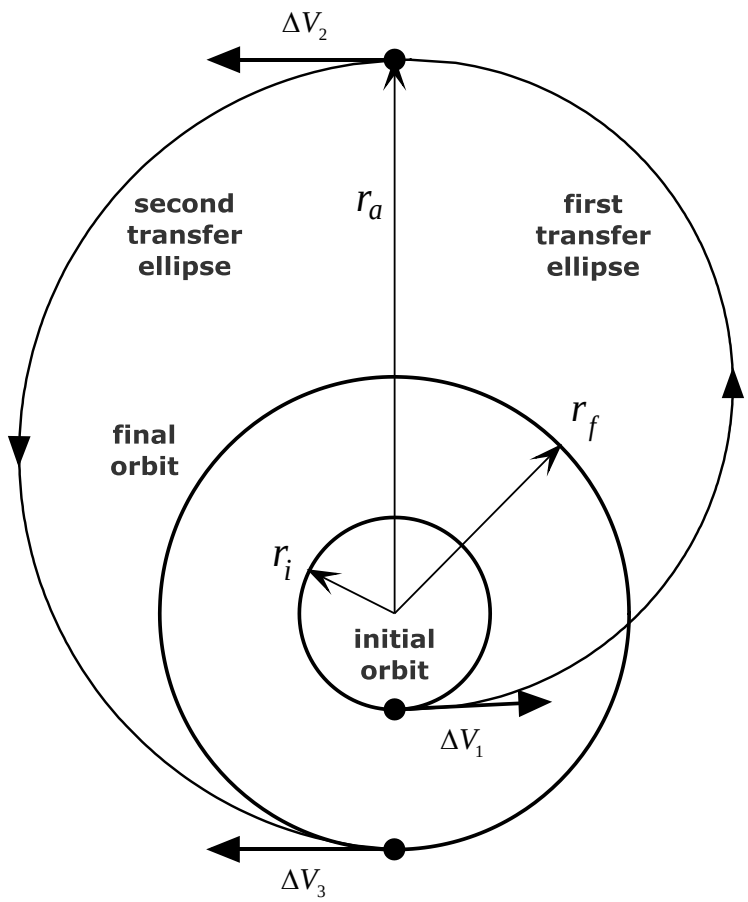
Here's the Hohmann transfer solution for this example. Since the radius ratio for this example is greater than 15.58, the bi-elliptic orbit transfer is more efficient than the Hohmann transfer.

```
Hohmann Orbit Transfer Analysis
-------------------------------

initial orbit altitude              300.0000 kilometers

initial orbit radius               6678.1363 kilometers

initial orbit inclination             0.0000 degrees

initial orbit velocity             7725.7606 meters/second

final orbit altitude              100000.0000 kilometers

final orbit radius                106378.1363 kilometers

final orbit inclination               0.0000 degrees

final orbit velocity               1935.7207 meters/second

first inclination change              0.0000 degrees

second inclination change             0.0000 degrees

total inclination change              0.0000 degrees

first delta-v                      2872.5124 meters/second

second delta-v                     1270.3893 meters/second

total delta-v                      4142.9017 meters/second

transfer orbit semimajor axis     56528.1363 kilometers
```

```
transfer orbit eccentricity       0.88186173

transfer orbit inclination            0.0000 degrees

transfer orbit perigee velocity   10598.2730 meters/second

transfer orbit apogee velocity      665.3314 meters/second

transfer orbit coast time         66877.1857 seconds
                                   1114.6198 minutes
                                     18.5770 hours
```

Technical Discussion

The following diagram (not to scale) illustrates the geometry of the coplanar bi-elliptic orbit transfer. In this figure, r_i is the geocentric radius of the initial circular orbit, r_a is the apogee radius of the two transfer ellipses, and r_f is the radius of the final circular orbit. The locations and directions of the first, second and third impulsive maneuvers are labeled $\Delta V_1, \Delta V_2$ and ΔV_3, respectively.



The total impulsive delta-v for a bi-elliptic orbital transfer is a function of the initial, intermediate and final orbital altitudes. The relationship between geocentric radius and orbital altitude is as follows:

$$r_i = r_e + h_i \qquad r_a = r_e + h_a \qquad r_f = r_e + h_f$$

where r_i is the geocentric radius of the initial circular park orbit, r_a is the radius at the intermediate impulse, and r_f is the radius of the final circular mission orbit. In these equations, h_i, h_a and h_f are the corresponding altitudes and r_e is the radius of the Earth.

The magnitude of the first impulse is

$$\Delta V_1 = v_{p_1} - v_i$$

and is simply the difference between the speed on the initial circular orbit and the perigee speed of the first transfer ellipse. The scalar magnitude of the second impulse is

$$\Delta V_2 = v_{2_a} - v_{1_a}$$

which is the difference between the speed at apogee of the first transfer ellipse and the apogee speed of the second transfer ellipse.

Finally, the scalar magnitude of the final delta-v is

$$\Delta V_3 = v_f - v_{2_p}$$

which is the speed difference between the final circular orbit and the speed at perigee of the second transfer ellipse.

The orbital speeds required for these computations can be determined from

$$v_i = \sqrt{\frac{\mu}{r_i}} \qquad v_f = \sqrt{\frac{\mu}{r_f}}$$

$$v_{p_1} = \sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_1}} \qquad v_{a_1} = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a_1}}$$

$$v_{p_2} = \sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_2}} \qquad v_{a_2} = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a_2}}$$

In these equations, μ is the gravitational constant of the central body, and the semimajor axis of each transfer ellipse is computed from

$$a_1 = \frac{r_i + r_a}{2} \qquad a_2 = \frac{r_a + r_f}{2}$$

The transfer time from the first impulse to the final impulse is equal to the sum of the half orbital periods of the two transfer ellipses according to

$$\tau = \pi\sqrt{\frac{a_1^3}{\mu}} + \pi\sqrt{\frac{a_2^3}{\mu}}$$

Software implementation

For the `optimal intermediate altitude` script option, the software calls the built-in bounded minimization MATLAB algorithm to solve for the intermediate apogee altitude that minimizes the total delta-v required for the mission.

The call to the algorithm is as follows:

```
[x, fx, exitflag] = fminbnd('befunc', xmin, xmax);
```

where `befunc` is the objective function for this problem. In the argument list, `xmin` and `xmax` are the lower and upper bounds for the intermediate radius, `x` is the solved-for intermediate altitude, and `fx` is the corresponding total delta-v. In this script, they are equal to the radius of the final orbit and one hundred times this radius as follows

```
xmin = rf;

xmax = 100.0 * rf;
```

The following is the MATLAB source code for the objective function.

```
function fx = befunc (x)

% bi-elliptic radius objective function

% required by bielliptic.m

% Orbital Mechanics with MATLAB

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

global mu ri rf

% semimajor axes of the transfer orbits (kilometers)

sma1 = (ri + x) / 2.0;

sma2 = (x + rf) / 2.0;

% initial orbit velocity (kilometers/second)

vi = sqrt(mu / ri);

% first transfer ellipse periapsis velocity (kilometers/second)

vt1a = sqrt((2.0 * mu / ri) - (mu / sma1));

% first transfer ellipse apoapsis velocity (kilometers/second)

vt1b = sqrt((2.0 * mu / x) - (mu / sma1));

% second transfer ellipse periapsis velocity (kilometers/second)

vt2b = sqrt((2.0 * mu / x) - (mu / sma2));
```

```
% second transfer ellipse periapsis velocity (kilometers/second)

vt2c = sqrt((2.0 * mu / rf) - (mu / sma2));

% final orbit velocity (kilometers/second)

vf = sqrt(mu / rf);

% compute delta-v contibutions (kilometers/second)

dva = abs(vt1a - vi);

dvb = abs(vt2b - vt1b);

dvc = abs(vf - vt2c);

% calculate objective function value

fx = dva + dvb + dvc;
```

The classic bi-elliptic technical paper is "The Bi-elliptical Transfer Between Co-planar Circular Orbits" by Rudolf F. Hoelker and Robert Silber which was published in *Advances in Ballistic Missiles and Space Technology*, Volume 3, Pergamon, Oxford, 1961.

An excellent document that describes impulsive orbital transfers is "Optimal Impulsive Maneuvers in Orbital Transfers" by Silvano Sgubini and Paolo Teofilatto. A PDF version of this document can be downloaded from http://naca.central.cranfield.ac.uk/dcsss/2002/E05a_sgubini.pdf.

Additional information can also be found in Chapter 6 of "Fundamentals of Astrodynamics and Applications" by David A. Vallado, Microcosm Press, 2007.

Primer Vector Analysis

This section summarizes the primer vector analysis included with this MATLAB script. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Jezewski.

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer is a unit vector aligned with the impulse and has unit magnitude $\left(\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T \text{ and } \|\mathbf{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\mathbf{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final times) $\mathbf{p} \bullet \dot{\mathbf{p}} = 0$; therefore, $d\|\mathbf{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitudes of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the impulse times. These four cases for non-zero slopes are summarized as follows;

- If $\dot{p}_0 > 0$ and $\dot{p}_f < 0 \rightarrow$ perform an initial coast before the first impulse and add a final coast after the second impulse
- If $\dot{p}_0 > 0$ and $\dot{p}_f > 0 \rightarrow$ perform an initial coast before the first impulse and move the second impulse to a later time
- If $\dot{p}_0 < 0$ and $\dot{p}_f < 0 \rightarrow$ perform the first impulse at an earlier time and add a final coast after the second impulse
- If $\dot{p}_0 < 0$ and $\dot{p}_f > 0 \rightarrow$ perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.

First partition the two-body state transition matrix as follows:

$$\Phi(t,t_0) = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any time *t* along a two body trajectory is given by

$$\mathbf{p}(t) = \Phi_{11}(t,t_0)\mathbf{p}_0 + \Phi_{12}(t,t_0)\dot{\mathbf{p}}_0$$

and the value of the primer vector derivative is

$$\dot{\mathbf{p}}(t) = \Phi_{21}(t,t_0)\mathbf{p}_0 + \Phi_{22}(t,t_0)\dot{\mathbf{p}}_0$$

which can also be expressed as

$$\begin{Bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{Bmatrix} = \Phi\left(t, t_0\right) \begin{Bmatrix} \mathbf{p}_0 \\ \dot{\mathbf{p}}_0 \end{Bmatrix}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\mathbf{p}\left(t_0\right) = \mathbf{p}_0 = \frac{\Delta \mathbf{V}_0}{\left|\Delta \mathbf{V}_0\right|} \qquad \mathbf{p}\left(t_f\right) = \mathbf{p}_f = \frac{\Delta \mathbf{V}_f}{\left|\Delta \mathbf{V}_f\right|}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the corresponding impulse.

The value of the primer vector derivative at the initial time is

$$\dot{\mathbf{p}}\left(t_0\right) = \dot{\mathbf{p}}_0 = \Phi_{12}^{-1}\left(t_f, t_0\right)\left\{\mathbf{p}_f - \Phi_{11}\left(t_f, t_0\right)\mathbf{p}_0\right\}$$

provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\left\|\mathbf{p}\right\|}{dt} = \frac{d}{dt}\left(\mathbf{p} \bullet \mathbf{p}\right)^2 = \frac{\dot{\mathbf{p}} \bullet \mathbf{p}}{\left\|\mathbf{p}\right\|}$$

As noted by D. J. Jezewski, the primer vector is sometimes called the Lagrange multiplier, costate vector or perhaps an adjoint variable.