

Discrete Orthogonal Polynomial Toolbox

DOPBox:

Investigation of Basis Function Quality

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Original: January 9, 2013
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Last Modified: May 28, 2013

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Abstract

This file uses the DOP Toolbox to investigate objective measures for the quality of discrete orthonormal basis functions \mathbf{B} . different measures are compared: the Gramian i.e, the determinant of the Gram matrix, $|\mathbf{G}| = |\mathbf{B}^T \mathbf{B}|$; the maximum error in the projection onto the orthogonal complement, $\max(\mathbf{I} - \mathbf{G})$; and the Frobenius norm of the projection onto the orthogonal complement $\|\mathbf{I} - \mathbf{G}\|_F$.

The quality of the new synthesis algorithm is verified.

1 Functions used from the `DOPBox` toolbox.

This file uses the functions:

1. `dop`: to synthesize the orthogonal basis functions.

2 Introduction

Much has been written on the application of discrete orthogonal basis functions, particularly in image processing [5, 9, 10, 8, 4, 11, 12, 1]. Unfortunately, very little work has been done to establish objective measures for the quality of basis functions. In general the quality of the basis functions was measured using the reconstruction quality of an image. This is unsatisfactory since the result is a mixture of the information content of the image and the quality of the basis functions.

Gram polynomials [3, 2, 6] are an interesting set of basis functions since they form an orthonormal basis function set. They are used in this investigation.

This file investigates three possible measures for the quality of basis functions. The three measures are then used to compare the quality of Gram polynomials using the method originally proposed by Gram which has become known as Gram-Schmidt orthogonalization, with the polynomial synthesis method proposed in [6] and applied in [7].

3 Proposed Measures of Quality

Objective measures of quality of a set of basis functions are required, if we wish to evaluate new synthesis methods.

Given a set of basis functions \mathbf{B} formed by concatenating the individual basis functions as the columns of \mathbf{B} , the Gram matrix is defined as,

$$\mathbf{G} = \mathbf{B}^T \mathbf{B}. \quad (1)$$

Ideally, the Gram matrix should be the identity matrix independent of the degree of the set of basis functions. Consequently, the projection onto the orthogonal complement,

$$\mathbf{G}^\perp = \mathbf{I} - \mathbf{G} = \mathbf{0} \quad (2)$$

should be exactly the zero matrix.

3.1 The Gramian as a measure of quality

The Gramian is defined as the determinant of the Gram matrix. Here we define the symbol $g_B \triangleq |\mathbf{G}| = |\mathbf{B}^T \mathbf{B}|$ as the Gramian of the set of basis functions contained in \mathbf{B} . The determinant of the Gram matrix should be exactly, i.e. $|\mathbf{G}| = 1$. Consequently the error measure can be defined as $\epsilon_g = 1 - g_B$.

3.2 The maximum in \mathbf{G}^\perp

The first new measure proposed here is the maximum value in the matrix $\epsilon_m = \max(\mathbf{G}^\perp)$.

3.3 The Frobenius norm of \mathbf{G}^\perp

The Frobenius norm of $\epsilon_F = \|\mathbf{G}^\perp\|_F$ is a measure for the total error.

3.4 Error Measures and number of Significant Digits

Ideally the error measures ϵ should be exactly zero. However, all computer systems have finite precision, e.g. MATLAB has `eps` = 2.2204e-16. This is the smallest relative distance between two numbers, if we consider the number 1 there there are approximately $d_s = 16$ significant digits. The significant digits for the basis functions can be estimates from the error measures as follows, $d_s \approx \log_{10}(\epsilon)$

4 A new Synthesis Algorithm

The synthesis method proposed by Gram [3] is now known as Gram Schmidt orthogonalization. This method is, however, known to be numerically unstable. A method based on complete re-orthogonalization was proposed in [6] and applied to inverse problems in [7]. The proposed error measures are used here to compare the two synthesis methods.

5 Matlab Code

A few preparatory lines of code.

```
1 close all;
2 clear all;
3 setUpGraphics;
```

define the minimum and maximum degrees to be tested

```
4 minD = 5;
5 maxD = 70;
6 %
7 % Define a vector of degrees
8 %
9 d = minD : maxD ;
10 %
```

```

11 % Prepare storage for the results
12 %
13 noSims = length( d );
14 Eg = zeros( noSims, 1 );
15 Em = zeros( noSims, 1 );
16 Ef = zeros( noSims, 1 );

Compute the measures of error for each degree

17 for k=1:noSims
18     %
19     % Synthesize the basis functions using Gram Schmidt orthogonalization
20     %
21     B2 = dopGram( d(k) );
22     %
23     % Compute the Gram matrix and its orthogonal complement
24     %
25     G = B2' * B2;
26     Gort = G - eye(d(k));
27     %
28     % compute the error measures
29     %
30     Eg(k) = 1 - det( G );
31     Em(k) = max(abs(Gort(:)));
32     Ef(k) = norm(Gort, 'fro');
33     %
34 end;

```

5.1 Measures for Gram Schmidt synthesis

The error measures for the Gram Schmidt synthesis of the Gram polynomials are shown in Figure 5.1. The error measures ϵ_m and ϵ_f deliver similar results indicating that there is a low in significant digits even for very modest degrees and that the quality of the basis functions degenerates progressively. The Gramian ϵ_m shown no initial degradation up to a degree of $d \approx 30$ and then degenerates more rapidly. At degree $d \approx 60$ the basis functions have degenerated to an extent that there are no significant digits.

```

35 fig1 = figure;
36 plot( d, log10(abs(Eg)+eps), 'k' );
37 hold on;
38 plot( d, log10(Em), 'r' );
39 plot( d, log10(Ef), 'b' );
40 %
41 range = axis;
42 plot( [34,34], range(3:4), 'k' );
43 plot( [59,59], range(3:4), 'k' );
44 grid on;
45 %
46 xlabel( 'Degree d' );

```

```

47 ylabel( ' \log_{10}( \epsilon ) ' );
48 legend( ' \epsilon_g ', ' \epsilon_m ', ' \epsilon_f ', 'Location', 'NorthWest' );

```

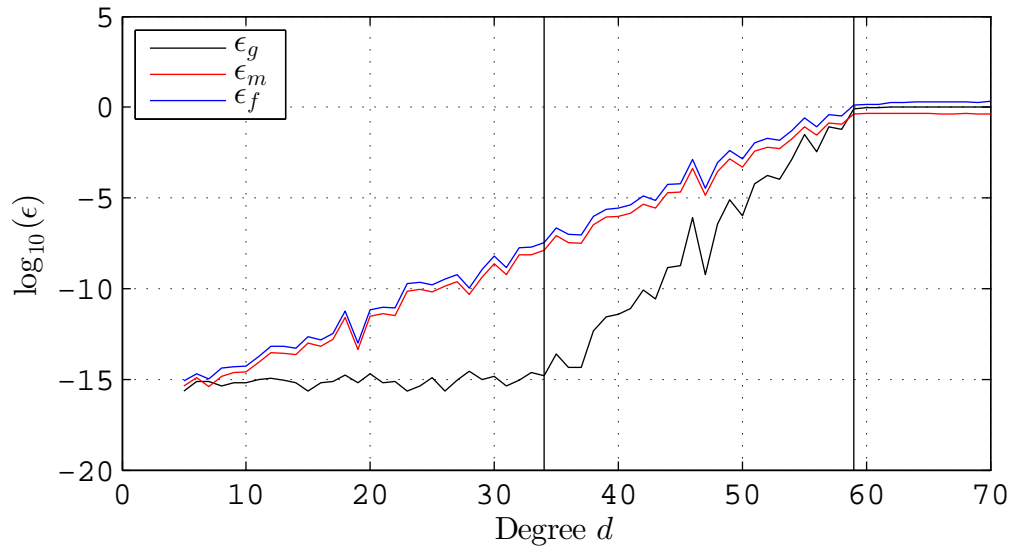


Figure 1: Error measures for the Gram Schmidt synthesis of the Gram polynomials.

5.2 Measures for synthesis using `dop.m`

In this the synthesis algorithm with complete orthogonalization is investigated. It produces high quality basis functions for very high degrees. To avoid inordinate computation time the degrees at which the results are logarithmically spaced, here between $d = 10 \dots 1000$.

```

49
50 % setup the vector of degrees
51 %
52 Log10minD = 1;
53 Log10maxD = 3;
54 %
55 % Use a logarithmic spacing of the degree to save time
56 %
57 d = round(logspace( Log10minD, Log10maxD ));
58 %
59 % Prepare storage for the results
60 %
61 noSims = length( d );
62 Eg = zeros( noSims, 1 );
63 Em = zeros( noSims, 1 );
64 Ef = zeros( noSims, 1 );

```

Compute the measures of error for each degree

```

65 for k=1:noSims

```

```

66     %
67     % Synthesize the basis functions using the new procedure
68     %
69     B2 = dop( d(k) );
70     %
71     % Compute the Gram matrix and its orthogonal complement
72     %
73     G = B2' * B2;
74     Gort = G - eye(d(k));
75     %
76     % compute the error measures
77     %
78     Eg(k) = 1 - det( G );
79     Em(k) = max(abs(Gort(:)));
80     Ef(k) = norm(Gort, 'fro');
81     %
82 end;

```

Plot the results for the new algorithm

```

83 fig2 = figure;
84 plot( d, log10(abs(Eg)+eps), 'k' );
85 hold on;
86 plot( d, log10(Em), 'r' );
87 plot( d, log10(Ef), 'b' );
88 grid on;
89 %
90 xlabel( 'Degree d' );
91 ylabel( ' \log_{10}( \epsilon ) ');
92 legend( '\epsilon_g', '\epsilon_m', '\epsilon_f', 'Location', 'NorthWest' );

```

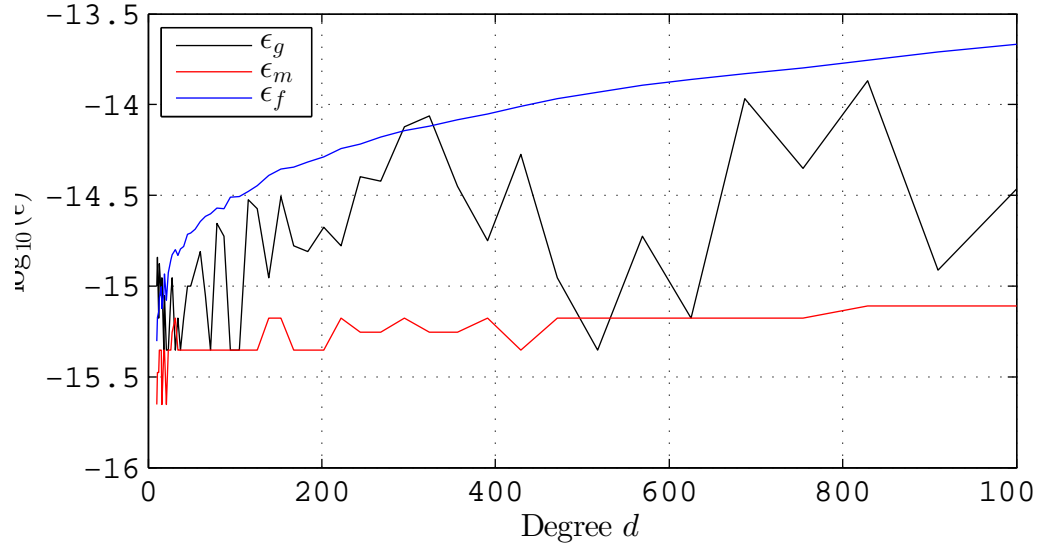


Figure 2: Error measures for the complete orthogonalization synthesis of the Gram polynomials.

Note that there are still more than $d_s > 13$ significant digits at degree $d = 1000$. These basis functions are for all intents and purposes free from error.

6 Conclusions

The Frobenius norm of the orthogonal complement of the Gram matrix yields a stable estimate for the error in a set of orthogonal basis functions. It corresponds to the upperbound of the deviation of the Gramian from 1, it is however, more stable (see Figure 5.2).

The synthesis method for basis functions implemented in `dop.m` yield basis functions which have excellent quality even at degrees of $d = 1000$.

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