Low Thrust Transfer Between Non-coplanar Circular Orbits

This document describes a MATLAB script named `ltot.m` that can be used to determine the characteristics of low-thrust orbital transfer between two non-coplanar circular orbits. The algorithm used in this script is based on Edelbaum's method which is described in Chapter 14 of the book *Orbital Mechanics* by V. Chobotov and the technical paper, "The Reformulation of Edelbaum's Low-thrust Transfer Problem Using Optimal Control Theory" by J. A. Kechichian, AIAA-92-4576-CP. The original Edelbaum algorithm is described in "Propulsion Requirements for Controllable Satellites", ARS Journal, August 1961, pages 1079-1089.

This algorithm is valid for total inclination changes Δi given by $0 < \Delta i < 114.6^\circ$. This numerical method assumes that the thrust acceleration magnitude is constant during the orbit transfer. It also assumes that the magnitude of the thrust acceleration is low enough to include the assumption that the intermediate orbits of the transfer are circular.

Interacting with the script

This MATLAB script will prompt you for the initial and final altitudes and orbital inclinations, the initial right ascension of the ascending node (RAAN) and the thrust acceleration. The following is a typical user interaction with this script. It illustrates an orbital transfer from a low Earth orbit (LEO) with an inclination of 28.5° to a geosynchronous Earth orbit (GSO) with an orbital inclination of 5°. The thrust acceleration for this example is `0.000350` meters/second^2 and the initial right ascension of the ascending node (RAAN) is equal to 10°.

```
Low-thrust Orbit Transfer Analysis


please input the initial altitude (kilometers)
? 621.86

please input the final altitude (kilometers)
? 35787.86

please input the initial orbital inclination (degrees)
(0 <= inclination <= 180)
? 28.5

please input the final orbital inclination (degrees)
(0 <= inclination <= 180)
? 5.0

please input the initial RAAN (degrees)
(0 <= RAAN <= 360)
? 10.0

please input the thrust acceleration (meters/second^2)
? 3.5e-4
```

The following is the output created by this MATLAB script for this example.

```
   Low-thrust Orbit Transfer Analysis

initial orbit altitude       621.8600 kilometers
```

```
initial orbit inclination      28.5000 degrees

initial RAAN                   10.0000 degrees

initial orbit velocity       7546.0553 meters/second


final orbit altitude        35787.8600 kilometers

final orbit inclination         5.0000 degrees

final orbit velocity         3074.5935 meters/second

total inclination change       23.5000 degrees

total delta-v                5412.5496 meters/second

thrust duration               178.9864 days

initial yaw angle              19.9486 degrees

thrust acceleration           0.350000 meters/second^2
```

The `ltot` MATLAB script will also numerically integrate the two-body equations of motion using the analytic Edelbaum solution for the yaw angle. The following is a summary of the final classical orbital elements and state vector for this example.

```
final classical orbital elements and state vector - integrated solution
-----------------------------------------------------------------------

        sma (km)             eccentricity        inclination (deg)         argper (deg)
 +4.21660712777590e+04  +1.72288028583874e-03  +4.98587779240600e+00  +1.29690954287292e+02

       raan (deg)        true anomaly (deg)         arglat (deg)           period (hrs)
 +1.03538378105804e+01  +9.04217700516484e+01  +2.20112724338940e+02  +2.39360888197177e+01

        rx (km)                 ry (km)                 rz (km)                rmag (km)
 -2.68586395397592e+04  -3.24199106724504e+04  -2.36114057407478e+03  +4.21664808911880e+04

        vx (kps)                vy (kps)                vz (kps)               vmag (kps)
 +2.36628844277148e+00  -1.95236286691198e+00  -2.04653260319509e-01  +3.07456089799016e+00
```

The following is a brief summary of the information provided by this MATLAB script.

sma (km) = semimajor axis in kilometers

eccentricity = orbital eccentricity (non-dimensional)

inclination (deg) = orbital inclination in degrees

argper (deg) = argument of perigee in degrees

raan (deg) = right ascension of the ascending node in degrees

true anomaly (deg) = true anomaly in degrees

arglat (deg) = argument of latitude in degrees. The argument of latitude is the sum of true anomaly and argument of perigee.

period (hrs) = orbital period in hours

rx (km) = x-component of the position vector in kilometers

ry (km) = y-component of the position vector in kilometers

rz (km) = z-component of the position vector in kilometers

rmag (km) = scalar magnitude of the position vector in kilometers

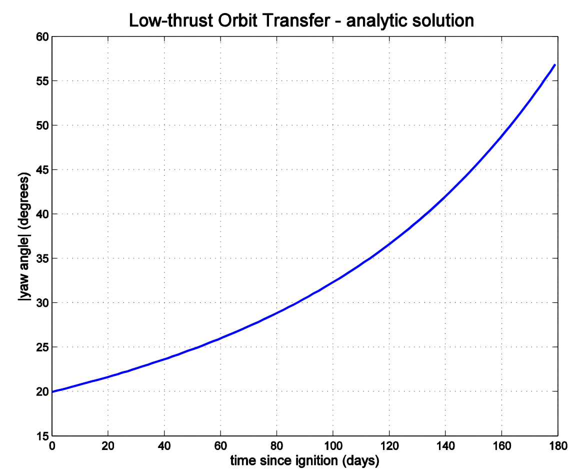
vx (km/sec) = x-component of the velocity vector in kilometers per second

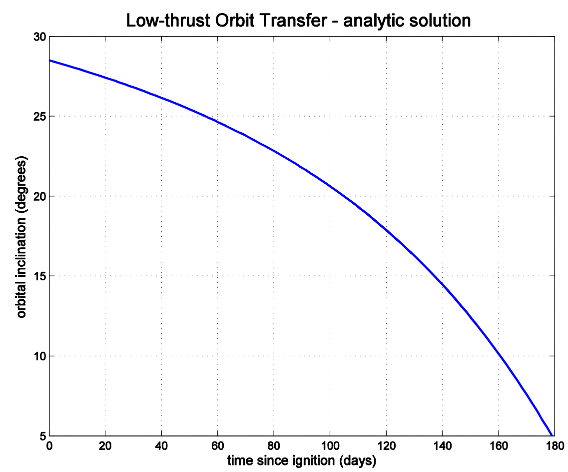
vy (km/sec) = y-component of the velocity vector in kilometers per second

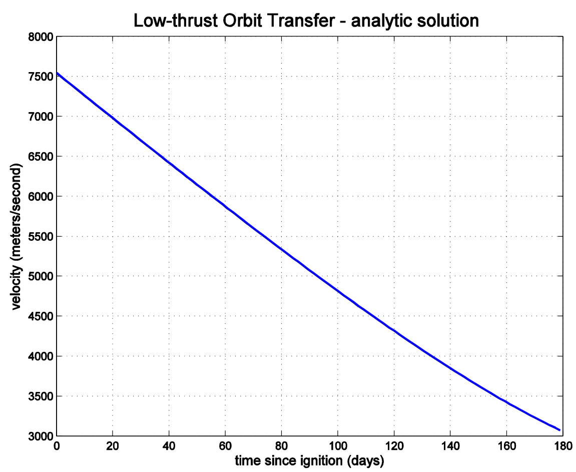
vz (km/sec) = z-component of the velocity vector in kilometers per second

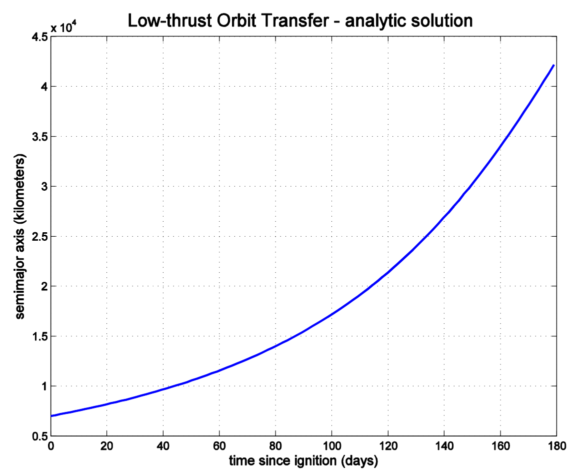
vmag (km/sec) = scalar magnitude of the velocity vector in kilometers per second

The software will also graphically display the time evolution of the absolute value of the yaw angle, orbital inclination, velocity and semimajor axis for the *analytic* Edelbaum solution. The graphics for this example are as follows:

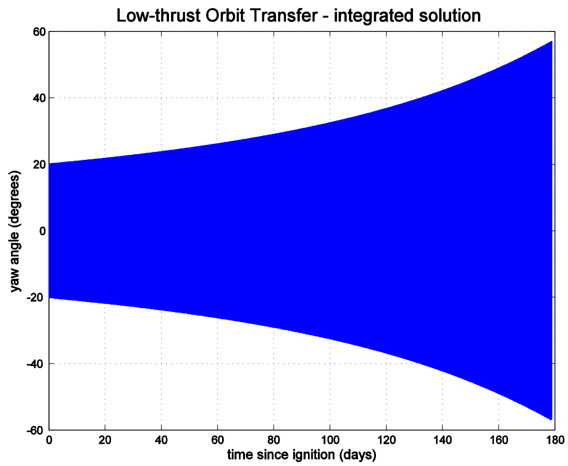


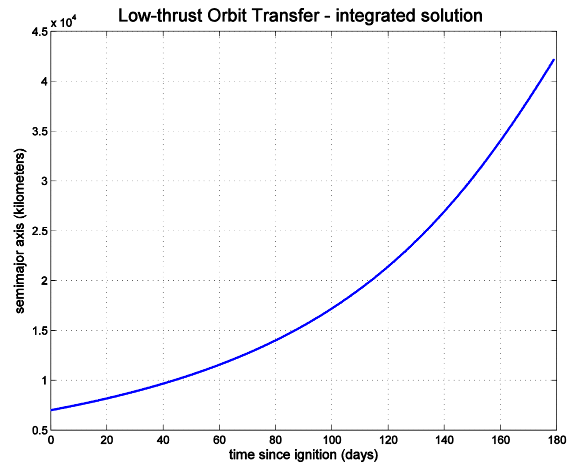


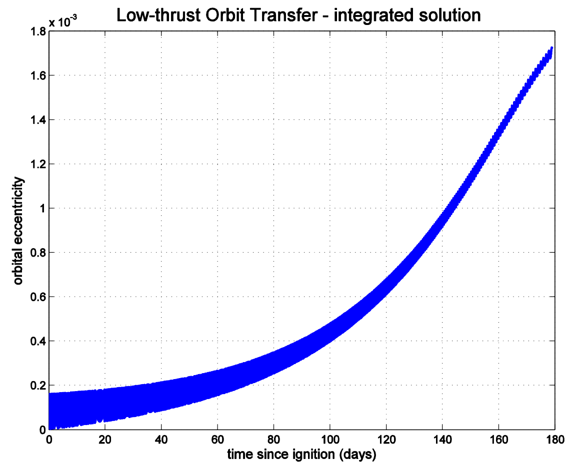


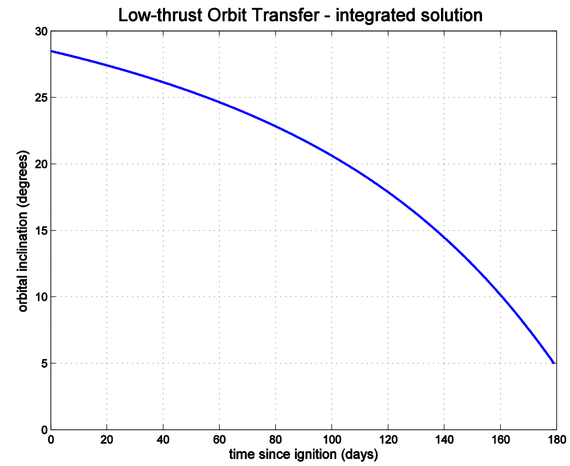


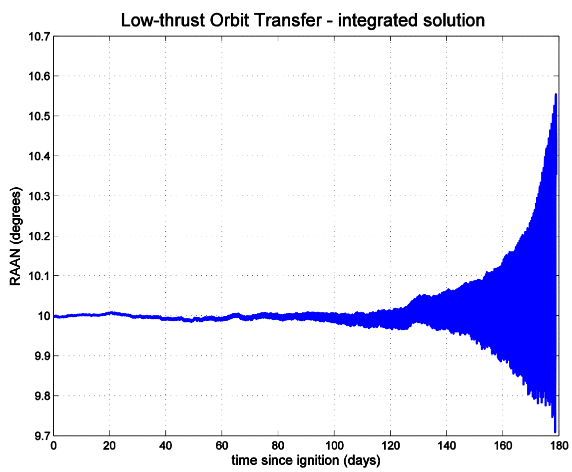
Finally, the `ltot` MATLAB script will graphically display the numerically integrated solution. The following are plots of the yaw angle, semimajor axis, orbital eccentricity, orbital inclination, RAAN and a three-dimensional display of the transfer orbit.

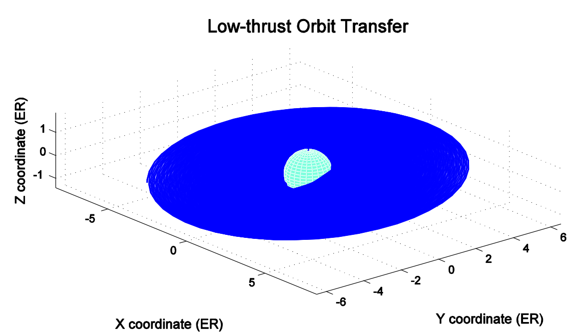












From these displays, we can see that the evolution of the orbital eccentricity remains small, thus validating the assumption that the transfer orbit remains nearly circular. Furthermore, since the RAAN remains basically constant, the yaw steering only modifies the orbital inclination.

Technical discussion

According to Edelbaum's analytic solution, the initial thrust vector yaw angle β_0 is given by the following expression

$$\tan\beta_0 = \frac{\sin\left(\dfrac{\pi}{2}\Delta i\right)}{\dfrac{V_0}{V_f} - \cos\left(\dfrac{\pi}{2}\Delta i\right)}$$

where the speed on the initial circular orbit is $V_0 = \sqrt{\mu/r_0}$ and the speed on the final circular orbit is $V_f = \sqrt{\mu/r_f}$. In these equations $r_0 = r_e + h_0$ is the geocentric radius of the initial orbit, $r_f = r_e + h_f$ is the geocentric radius of the final orbit, r_e is the radius of the Earth and μ is the gravitational constant of the Earth. The initial altitude is h_0 and the final altitude is h_f.

The total velocity change required for a low-thrust orbit transfer is given by

$$\Delta V = V_0\cos\beta_0 - \frac{V_0\sin\beta_0}{\tan\left(\dfrac{\pi}{2}\Delta i + \beta_0\right)} = \sqrt{V_0^2 - 2V_0V_f\cos\left(\dfrac{\pi}{2}\Delta i\right) + V_f^2}$$

The total transfer time is given by $t = \Delta V/f$ where f is the thrust acceleration. The time evolution of the yaw angle, speed and inclination change are given by the following three expressions:

$$\beta(t) = \tan^{-1}\left(\frac{V_0\sin\beta_0}{V_0\cos\beta_0 - f\,t}\right)$$

$$V(t) = \sqrt{V_0^2 - 2V_0 f\,t\cos\beta_0 + f^2t^2} = V_0\sin\beta_0\frac{\sqrt{1+\tan^2\beta}}{\tan\beta}$$

$$\Delta i(t) = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{f\,t - V_0\cos\beta_0}{V_0\sin\beta_0}\right) + \frac{\pi}{2} - \beta_0\right]$$

Modified equinoctial orbital elements

The modified equinoctial orbital elements are a set of orbital elements that are useful for trajectory analysis and optimization. They are valid for circular, elliptic, and hyperbolic orbits. These equations exhibit no singularity for zero eccentricity and orbital inclinations equal to 0 and 90 degrees. However, two components of the orbital element set are singular for an orbital inclination of 180 degrees.

The relationship between direct modified equinoctial and classical orbital elements is defined by the following definitions

$$p = a\left(1-e^2\right) \qquad f = e\cos\left(\omega+\Omega\right) \qquad g = e\sin\left(\omega+\Omega\right)$$

$$h = \tan\left(i/2\right)\cos\Omega \qquad k = \tan\left(i/2\right)\sin\Omega \qquad L = \Omega+\omega+\theta$$

where

$p =$ semiparameter
$a =$ semimajor axis
$e =$ orbital eccentricity
$i =$ orbital inclination
$\omega =$ argument of periapsis
$\Omega =$ right ascension of the ascending node
$\theta =$ true anomaly
$L =$ true longitude

The relationship between classical and modified equinoctial orbital elements is:

semimajor axis $$a = \frac{p}{1-f^2-g^2}$$

orbital eccentricity $$e = \sqrt{f^2+g^2}$$

orbital inclination $$i = 2\tan^{-1}\left(\sqrt{h^2+k^2}\right)$$

argument of periapsis $$\omega = \tan^{-1}\left(g/f\right) - \tan^{-1}\left(k/h\right)$$

right ascension of the ascending node $$\Omega = \tan^{-1}\left(k/h\right)$$

true anomaly $$\theta = L - \left(\Omega+\omega\right) = L - \tan^{-1}\left(g/f\right)$$

The mathematical relationships between an inertial state vector and the corresponding modified equinoctial elements are summarized as follows:

position vector

$$\mathbf{r} = \begin{bmatrix} \dfrac{r}{s^2}\left(\cos L + \alpha^2\cos L + 2hk\sin L\right) \\ \dfrac{r}{s^2}\left(\sin L - \alpha^2\sin L + 2hk\cos L\right) \\ \dfrac{2r}{s^2}\left(h\sin L - k\cos L\right) \end{bmatrix}$$

velocity vector

$$\mathbf{v} = \begin{bmatrix} -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(\sin L + \alpha^2 \sin L - 2hk\cos L + g - 2fhk + \alpha^2 g\right) \\ -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(-\cos L + \alpha^2 \cos L + 2hk\sin L - f + 2ghk + \alpha^2 f\right) \\ \dfrac{2}{s^2}\sqrt{\dfrac{\mu}{p}}\left(h\cos L + k\sin L + fh + gk\right) \end{bmatrix}$$

where

$$\alpha^2 = h^2 - k^2 \qquad s^2 = 1 + h^2 + k^2$$

$$r = \frac{p}{w} \qquad w = 1 + f\cos L + g\sin L$$

The system of first-order modified equinoctial equations of orbital motion are given by

$$\dot{p} = \frac{dp}{dt} = \frac{2p}{w}\sqrt{\frac{p}{\mu}}\,\Delta_t$$

$$\dot{f} = \frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left[\Delta_r \sin L + \left[(w+1)\cos L + f\right]\frac{\Delta_t}{w} - \left(h\sin L - k\cos L\right)\frac{g\Delta_n}{w}\right]$$

$$\dot{g} = \frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left[-\Delta_r \cos L + \left[(w+1)\sin L + g\right]\frac{\Delta_t}{w} - \left(h\sin L - k\cos L\right)\frac{f\Delta_n}{w}\right]$$

$$\dot{h} = \frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\cos L \qquad \dot{k} = \frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\sin L$$

$$\dot{L} = \frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \frac{1}{w}\sqrt{\frac{p}{\mu}}\left(h\sin L - k\cos L\right)\Delta_n$$

where $\Delta_r, \Delta_t, \Delta_n$ are *non-two-body* perturbations in the radial, tangential and normal directions, respectively. The radial direction is along the geocentric radius vector of the spacecraft measured positive in a direction away from the gravitational center, the tangential direction is perpendicular to this radius vector measured positive in the direction of orbital motion, and the normal direction is positive in the direction of the angular momentum vector of the spacecraft's orbit.

The equations of orbital motion can also be expressed in vector form as follows:

$$\dot{\mathbf{y}} = \frac{d\mathbf{y}}{dt} = \mathbf{A}(\mathbf{y})\mathbf{P} + \mathbf{b}$$

where

$$
\mathbf{A} = \begin{Bmatrix}
0 & \dfrac{2p}{w}\sqrt{\dfrac{p}{\mu}} & 0 \\
\sqrt{\dfrac{p}{\mu}}\sin L & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left\{(w+1)\cos L + f\right\} & -\sqrt{\dfrac{p}{\mu}}\dfrac{g}{w}\left\{h\sin L - k\cos L\right\} \\
-\sqrt{\dfrac{p}{\mu}}\cos L & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left\{(w+1)\sin L + g\right\} & \sqrt{\dfrac{p}{\mu}}\dfrac{f}{w}\left\{h\sin L - k\cos L\right\} \\
0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\cos L}{2w} \\
0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\sin L}{2w} \\
0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left\{h\sin L - k\cos L\right\}
\end{Bmatrix}
$$

and

$$
\mathbf{b} = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \sqrt{\mu p}\left(\frac{w}{p}\right)^2 \right]^T
$$

The total non-two-body acceleration vector is given by

$$
\mathbf{P} = \Delta_r \hat{\mathbf{i}}_r + \Delta_t \hat{\mathbf{i}}_t + \Delta_n \hat{\mathbf{i}}_n
$$

where $\hat{\mathbf{i}}_r, \hat{\mathbf{i}}_t$ and $\hat{\mathbf{i}}_n$ are unit vectors in the radial, tangential and normal directions. These unit vectors can be computed from the inertial position vector **r** and velocity vector **v** according to

$$
\hat{\mathbf{i}}_r = \frac{\mathbf{r}}{|\mathbf{r}|} \qquad \hat{\mathbf{i}}_n = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|} \qquad \hat{\mathbf{i}}_t = \hat{\mathbf{i}}_n \times \hat{\mathbf{i}}_r = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{|\mathbf{r}\times\mathbf{v}||\mathbf{r}|}
$$

For *unperturbed* two-body motion, $\mathbf{P} = 0$ and the first five equations of motion are simply $\dot{p} = \dot{f} = \dot{g} = \dot{h} = \dot{k} = 0$. Therefore, for two-body motion these modified equinoctial orbital elements are constant. The true longitude is often called the *fast variable* of this orbital element set.

Propulsive thrust

The acceleration due to propulsive thrust can be expressed as

$$
\mathbf{a}_T = \frac{T}{m(t)}\hat{\mathbf{u}}_T(t)
$$

where T is the constant thrust magnitude, m is the spacecraft mass and $\hat{\mathbf{u}}_T = \begin{bmatrix} u_{T_r} & u_{T_t} & u_{T_n} \end{bmatrix}^T$ is the unit pointing thrust vector expressed in the spacecraft-centered radial-tangential-normal coordinate system.

This MATLAB script assumes that the thrust acceleration is constant during the entire orbit transfer.

The components of the unit thrust vector in the modified equinoctial frame can also be defined in terms of the in-plane pitch angle β and the out-of-plane yaw angle ψ as follows:

$$u_{T_r} = \sin\beta \qquad u_{T_t} = \cos\beta\cos\psi \qquad u_{T_n} = \cos\beta\sin\psi$$

Finally, the pitch and yaw angles can be determined from the components of the unit thrust vector according to

$$\beta = \sin^{-1}\left(u_{T_r}\right) \qquad \psi = \tan^{-1}\left(u_{T_n}, u_{T_t}\right)$$

Both steering angles are defined with respect to a local-vertical, local-horizontal (LVLH) system located at the spacecraft. The in-plane pitch angle is positive above the "local horizontal" and the out-of-plane yaw angle is positive in the direction of the angular momentum vector. The inverse tangent calculation in the second equation is a four quadrant operation.

In this MATLAB script, the pitch angle is assumed to be zero during the entire orbit transfer. Furthermore, the yaw angle at any simulation time during the numerical integration is given by the Edelbaum analytic solution. However, to actually perform the inclination change, the sign of the absolute value of the yaw steering angle is switched at the antinodes of the transfer orbit.

The following is the MATLAB source code that performs the sign switching based on the current value of the argument of latitude (sum of argument of perigee and true anomaly). In this code, `beta_tmp` is the current absolute value of the yaw angle.

```
% compute yaw angle based on current orbital position (radians)

if (arglat >= 0.0 && arglat < 0.5 * pi)

    beta_wrk = -beta_tmp;

end

if (arglat > 0.5 * pi && arglat < pi)

    beta_wrk = beta_tmp;

end

if (arglat > pi && arglat < 1.5 * pi)

   beta_wrk = beta_tmp;

end

if (arglat > 1.5 * pi && arglat < 2.0 * pi)

   beta_wrk = -beta_tmp;

end
```

Algorithm Resources

"On the Equinoctial Orbital Elements", R. A. Brouke and P. J. Cefola, *Celestial Mechanics*, Vol. 5, pp. 303-310, 1972.

"A Set of Modified Equinoctial Orbital Elements", M. J. H. Walker, B. Ireland and J. Owens, *Celestial Mechanics*, Vol. 36, pp. 409-419, 1985.

An Introduction to the Mathematics and Methods of Astrodynamics, Richard H. Battin, AIAA Education Series, 1987.

Analytical Mechanics of Space Systems, Hanspeter Schaub and John L. Junkins, AIAA Education Series, 2003.

Spacecraft Mission Design, Charles D. Brown, AIAA Education Series, 1992.

Orbital Mechanics, Vladimir A. Chobotov, AIAA Education Series, 2002.

Modern Astrodynamics, Victor R. Bond and Mark C. Allman, Princeton University Press, 1996.

"Analysis of Kechichian's Optimal Control Solution to a Low-Thrust Orbit Transfer Problem", Suzannah Lynn Sandrik, M.S. Thesis, University of Illinois at Urbana-Champaign, 2002.

"Using Edelbaum's Method to Compute Low-Thrust Transfers with Earth-Shadow Effects", Craig A. Kluever, AIAA Journal of Guidance, Control, and Dynamics, Vol. 34, No. 1, January-February 2011.

"Analytic Representations of Optimal Low-Thrust Transfer in Circular Orbit', Jean A. Kechichian, Chapter 6, "*Spacecraft Trajectory Optimization*, Edited by Bruce A. Conway, Cambridge University Press, New York, NY, 2010.

"Minimum-Time Low-Earth Orbit to High-Earth Orbit Low-Thrust Trajectory Optimization", Kathryn F. Schubert and Anil V. Rao, AAS 13-926, American Astronautical Society, 2013.

"Using Sparse Nonlinear Programming to Compute Low Thrust Orbit Transfers", John T. Betts, *The Journal of the Astronautical Sciences*, Vol. 41, No. 3, July-September 1993, pp. 349-371.

"Equinoctial Orbit Elements: Application to Optimal Transfer Problems", Jean A. Kechichian, AIAA 90-2976, AIAA/AAS Astrodynamics Conference, Portland, OR, 20-22 August 1990.