

## Chapter 4

# T Puzzle

*A classic puzzle demonstrates complex arithmetic.*



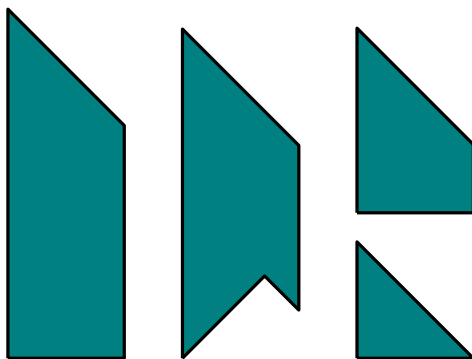
**Figure 4.1.** *The wooden T puzzle. Photo courtesy of Shop New Zealand, <http://www.shopnewzealand.co.nz>.*

I first saw the wooden T puzzle shown in figure 4.1 at Puzzling World in

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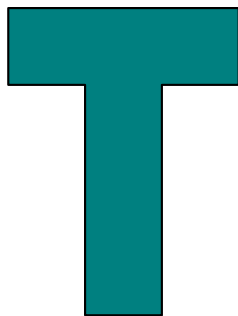
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Wanaka, New Zealand. They told me that it was their most popular puzzle. I have since learned that it was a well-known toy in the 1800s and an advertising tool in the early 1900s. The underlying mathematics involves geometry, trigonometry, and arithmetic with complex numbers. The `t_puzzle` program in the `exm` toolbox demonstrates some useful programming techniques.



**Figure 4.2.** *The four pieces.*

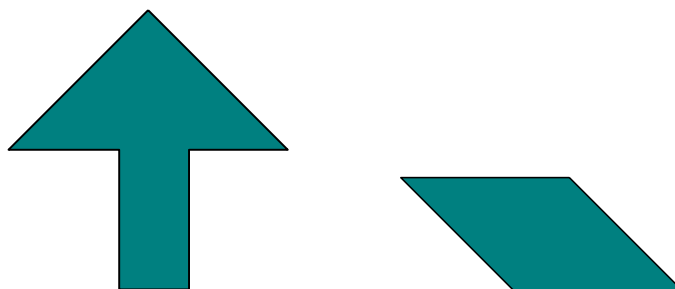
Figure 4.2 shows the electronic version of the four pieces. They all have the same width, but different heights. One of them has an unshapely chunk cut out of it, resulting in an irregular pentagon.



**Figure 4.3.** *The T.*

It turns out to be possible to arrange the four pieces to form the capital “T” shape shown in figure 4.3, as well as the arrow and the rhombus shapes shown in figure 4.4. What happened to those all of those  $45^\circ$  angles and what happened to that chunk?

If you do a Google search on “T-puzzle” you can quickly see how to solve the puzzle and form the T, but please try `t_puzzle` for a while before you go surfing for the solution. If you click near the center of one of the four pieces, you can move



**Figure 4.4.** *The arrow and the rhombus.*

it horizontally and vertically. If you click near one of the vertices, you can rotate a piece about its center. If you click with the right mouse button, or, on a one-button mouse, hold down the control key while you click, you can flip a piece over horizontally, changing its right/left-handed orientation. If you click in the subplot on the lower left of the window, you can cycle through images of the three shapes.

The key to effective computation involving the T-puzzle pieces is the use of complex arithmetic. A complex number is formed from a pair of real numbers and the imaginary unit  $i = \sqrt{-1}$ . For example,

$$z = 3 + 4i$$

The *real* part is 3 and the *imaginary* part is 4. This is the *Cartesian* representation of a complex number.

Addition of complex numbers simply involves addition of the real parts and of the imaginary parts.

$$\begin{aligned} & (3 + 4i) + (5 - i) \\ &= (3 + 5) + (4i - i) \\ &= 8 + 3i \end{aligned}$$

Multiplication of complex numbers is more interesting because it makes use of the fact that  $i^2 = -1$ .

$$\begin{aligned} & (3 + 4i) \cdot (5 - i) \\ &= (3 \cdot 5 + (4i) \cdot (-i)) + (3 \cdot (-i) + (4i) \cdot 5) \\ &= (15 + 4) + (-3 + 20)i \\ &= 19 + 17i \end{aligned}$$

A fundamental fact involving complex numbers is *Euler's formula*.

$$e^{i\phi} = \cos \phi + i \sin \phi$$

If you are not familiar with  $e$  or Euler's formula, see our chapter on the "Exponential Function" and the Wikipedia entry on "Euler's Identity". Or, you can just accept the formula as convenient notation.

Setting  $\phi = \pi$  leads to

$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi \\ &= -1 \end{aligned}$$

Moving the  $-1$  to the left hand side produces a beautiful equation involving five basic mathematical quantities,  $e$ ,  $i$ ,  $\pi$ ,  $1$ , and  $0$ .

$$e^{i\pi} + 1 = 0$$

The *polar form* of a complex number is

$$z = x + iy = re^{i\phi}$$

where

$$\begin{aligned} r &= |z| = \sqrt{x^2 + y^2} \\ x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$

Our T puzzle program uses the fact that multiplication by  $e^{i\theta}$  rotates a complex number through an angle  $\theta$ . To see this, let

$$\begin{aligned} w &= e^{i\theta} \\ z &= re^{i\phi} \end{aligned}$$

then

$$w \cdot z = e^{i\theta} \cdot re^{i\phi} = re^{i(\theta+\phi)}$$

In MATLAB the letter `i` can be used in any of three different roles. It can be an iteration counter,

```
for i = 1:10
```

or a subscript,

```
A(i,k)
```

or the imaginary unit.

```
z = 3 + 4i
```

The polar form of a complex number `z` is obtained with the MATLAB functions `abs(z)` and `angle(z)`. The quantity  $e^{i\phi}$  is written `exp(i*phi)`. For example

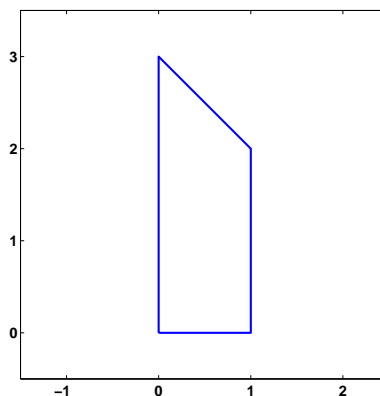
```
z = 3 + 4i
r = abs(z)
phi = angle(z)
w = r*exp(i*phi)
```

produces

```

z =
    3.0000 + 4.0000i
r =
    5
phi =
    0.9273
w =
    3.0000 + 4.0000i

```



**Figure 4.5.** *The complex coordinates of the vertices are  $0 + 0i$ ,  $1 + 0i$ ,  $1 + 2i$ , and  $0 + 3i$ .*

The largest of the four pieces of the T puzzle can be represented in MATLAB by the statement

```
z = [0 1 1+2i 3i 0]
```

The vector  $\mathbf{z}$  has five complex entries. The first two elements are 0 and 1; their imaginary parts are zero. The third element is  $1 + 2i$ ; its real part is 1 and its imaginary part is 2. After that comes  $3i$ ; its real part is zero and its imaginary part is 3. The last element of  $\mathbf{z}$  is a repeat of the first so that the line drawn by the statement

```
line(real(z), imag(z))
```

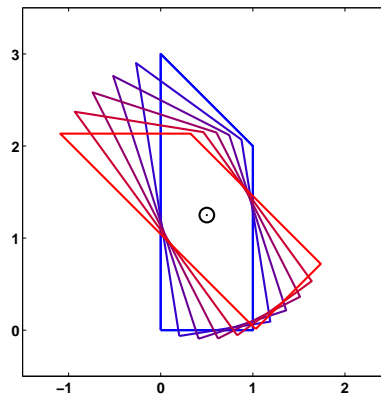
returns to the origin. The result is figure 4.5.

With this representation, the translations and rotations required by the graphics in `t_puzzle` can be programmed easily and efficiently. Translation is very easy. The statement

```
z = z - (3-i)/2
```

repositions the piece in figure 4.5 at one of the corners. Do you see which corner?

Rotations are also easily done. The statements



**Figure 4.6.** Rotating through multiples of nine degrees.

```
mu = mean(z(1:end-1))
theta = pi/20
omega = exp(i*theta)
z = omega*(z - mu) + mu
```

rotate the piece about its center through  $9^\circ$  in the counterclockwise direction. Figure 4.6 shows the result of repeating these statements several times. Let's look at each step more carefully. The statement

```
mu = mean(z(1:end-1))
```

drops the last element of  $z$  because it is a repeat of the first and then computes the complex average of the coordinates of the vertices. This gives the coordinates of what we can call the center of the polygon. The angle

$$\theta = \pi/20$$

is  $9^\circ$  in radians. The statements

```
omega = exp(i*theta)
z = omega*(z - mu) + mu
```

translate the piece so that its center is at the origin, do a complex scalar times vector multiplication to produce the rotation, and then translate the result back to its original center.

## Exercises

4.1 *Complex numbers.* Express each of these numbers in polar form.

$i$

$-4$   
 $3 - 4i$   
 $8 + 15i$

Express each of these numbers in Cartesian form.

$e^{-i\pi}$   
 $e^{i\pi/3}$   
 $e^i$   
 $e^3$

4.2 *Stop sign.* Try this for  $n = 8$  and for other small integer values of  $n$ . Describe and explain the results.

```

n = 8
z = exp(2*pi*i*(0:n)'/n)
plot(z, '-o')
axis square
s = sum(z)

```

4.3 *Strip.* How many different ways are there to form the shape in figure 4.7 with the T-puzzle pieces?



**Figure 4.7.** A strip.

4.4 *Area.* If the width of each of the T-puzzle pieces is one unit, what are their areas?

4.5 *Symmetry.* Which one of our three shapes – T, arrow and rhombus – does not have an axis of symmetry?

4.6 *Jumpy rotation.* Click near a vertex of one of the T-puzzle pieces and rotate the piece slowly. You should see that the rotation is not smooth, but proceeds in discrete jumps. Why? How large are the jumps? How does `t_puzzle` accomplish this?

4.7 *Snappy traslation.* Drag one of the T-puzzle pieces until it is close to, but not exactly touching, another. When you release the mouse button you sometimes see

the piece snap into place. Under what circumstances does this happen? How does `t_puzzle` accomplish it?

4.8 *Rotation*. Reproduce figure 4.6.

4.9 *Revelation*. Find `t_puzzle`'s hidden feature.

4.10 *Different puzzles*. Do a Google search on "T-puzzle". Include the quotes and hyphen in the search string so that you get an exact match. Some of the Web pages have pieces with different sizes than the ones we have described here.

- (a) How many different versions of the T-puzzle are there on the Web?
- (b) Can you make all three of our shapes – the T, arrow, and rhombus – with the pieces shown on these Web sites.
- (c) Modify our `t_puzzle` to use the set of pieces from one of the Web sites.