Dynamic Entropy Pooling: Portfolio Management with Views at Multiple Horizons

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Background

The profit-and-loss (P&L)

The market model

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Background

The standard approach to discretionary portfolio management (Black-Litterman, Entropy Pooling) processes subjective views that refer to the distribution of the market at a specific single investment horizon.

The standard approach to multi-period portfolio management with market impact (Garleanu-Pedersen) processes non-discretionary (systematic) signals.

Dynamic Entropy Pooling is a quantitative approach to perform dynamic portfolio management with discretionary, multi-horizon views.

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The profit-and-loss (P&L)

- We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:

\[ \Pi_{t+1} = b_t' \Delta X_{t+1} \]

- The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.
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Consider an equity share or an index. Then the risk driver is its log-value:
\[ X_t = \ln V_t \]

• The P&L of a portfolio with \( h_{n,t} \) shares in the \( n \)-th asset is:
\[ \Pi_{t+1} = \sum_n h_{n,t} V_{n,t} \times \left( \frac{V_{n,t+1}}{V_{n,t}} - 1 \right) \approx \sum_n b_{n,t} \Delta X_{n,t+1} \]

• More in general, in terms of a style/risk linear factor model:
\[ \Pi_{t+1} = \sum_k b_{k,t}^{\text{style}} \Delta X_{k,t+1}^{\text{style}} \]
• We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:

\[ \Pi_{t+1} = b'_t \Delta X_{t+1} \]

• The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.

• Suppose that the \( n \)-th asset is a fixed income instrument. Its value at the first order satisfies

\[ \Pi_{n,t+1} \approx - \sum_k dv01_{n,k,t} \Delta Y_{k,t+1} \]

where \( Y_{k,t} \) is the \( k \)-th key-rate on the yield curve; \( dv01_{n,k,t} \) is the dollar-sensitivity of the \( n \)-th instrument to \( Y_{k,t} \)

• Then the P&L due to a set of fixed income instruments is:

\[ \Pi_{t+1} \approx \sum_k \left( - \sum_n h_{n,t} dv01_{n,k,t} \right) \Delta X_{k,t+1} \]
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The profit-and-loss (P&L)

- We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:

\[ \Pi_{t+1} = b'_t \Delta X_{t+1} \]

- The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors, we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.

For a stock option, the risk drivers are the log-value of the underlying and the implied volatility \( X_t = \ln V_t \) and \( \sum^{impl} \)

- Then for a portfolio of stock options, the P&L is:

\[ \Pi_{t+1} \approx \sum_n h_{n,t} \delta_{n,t} V_{n,t} \Delta X_{n,t+1} + \sum_n h_{n,t} \nu_{n,t} \Delta \sum^{impl} \]

where \( \delta_{n,t} \) and \( \nu_{n,t} \) are the delta and vega of the \( n \)-th option.
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Consider a book of assets driven by a set of $\bar{n}$ risk drivers $\mathbf{X}_t$ (interest rates, implied volatility surfaces, log-prices, etc.)

We assume that the drivers follow a MVOU process:

$$d\mathbf{X}_t = (-\theta \mathbf{X}_t + \mathbf{\mu}) dt + \mathbf{\sigma} d\mathbf{W}_t$$

Choose a set of discrete monitoring dates $t, t + 1, \ldots, \bar{t}$

Stack the process at the monitoring times as follows:

$$\mathbf{X}_{t \sim \bar{t}} \equiv \begin{pmatrix} \mathbf{X}_t \\ \mathbf{X}_{t+1} \\ \vdots \\ \mathbf{X}_{\bar{t}} \end{pmatrix}$$

Then the process is jointly multivariate normal at all times

$$\mathbf{X}_{t \sim \bar{t}} | \mathbf{i}_t \sim N(\mathbf{\mu}_{t \sim \bar{t}}, \mathbf{\sigma}^2_{t \sim \bar{t}})$$
Market model

- The expectation vector of the is

\[
\mu_{t \rightarrow \tilde{t}} \equiv \begin{pmatrix}
    e^{-\theta} x_t + (\mathbb{1}_n - e^{-\theta}) \theta^{-1} \mu \\
    e^{-1\theta} x_t + (\mathbb{1}_n - e^{-1\theta}) \theta^{-1} \mu \\
    e^{-(\tilde{t} - t)\theta} x_t + (\mathbb{1}_n - e^{-(\tilde{t} - t)\theta}) \theta^{-1} \mu
\end{pmatrix}
\]

- The covariance matrix is

\[
\sigma^2_{t \rightarrow \tilde{t}} \equiv \begin{pmatrix}
    \sigma_0^2 & \sigma_0^2 e^{-\theta'} & \sigma_0^2 e^{-2\theta'} & \sigma_0^2 e^{-(\tilde{t} - t)\theta'} \\
    \sigma_0^2 e^{-\theta} & \sigma_1^2 & \sigma_1^2 e^{-\theta'} & \sigma_1^2 e^{-(\tilde{t} - t - 1)\theta'} \\
    \sigma_0^2 e^{-2\theta} & \sigma_1^2 e^{2\theta'} & \sigma_2^2 & \sigma_2^2 \\
    \sigma_0^2 e^{-(\tilde{t} - t)\theta} & \sigma_1^2 e^{-(\tilde{t} - t)\theta'} & \sigma_2^2 & \sigma_{t-t}^2
\end{pmatrix}
\]

where

\[
vec(\sigma^2_\tau) \equiv (\theta \oplus \theta)^{-1} (\mathbb{1}_n^2 - e^{-(\theta \oplus \theta)\tau}) vec(\sigma^2)
\]
We extend the Entropy Pooling approach in Meucci (2010) to the case of multiple horizons.

- **The prior**: assume a model for the joint distribution of the process at the monitoring times:
  \[ X_{t \sim \tilde{t}} | \mathbf{i}_t \sim f \]

- **The views**: are statements (constraints) on the yet-to-be defined distribution of the process:
  \[ g \in \mathcal{V}_t \]

- **The posterior**: is the closest distribution to the prior that satisfies the views:
  \[ \bar{f} \equiv \text{argmin}_{g \in \mathcal{V}_t} \{ \mathcal{E}(g, f) \} \]

where the “distance” is the relative entropy

\[ \mathcal{E}(g, f) \equiv \int g(x_t, \ldots, x_{\tilde{t}}) \ln \frac{g(x_t, \ldots, x_{\tilde{t}})}{f(x_t, \ldots, x_{\tilde{t}})} dx_t \cdots dx_{\tilde{t}} \]
We extend the Entropy Pooling approach in Meucci (2010) to the case of multiple horizons

• **The prior**: assume a MVOU model for the joint distribution of the process at the monitoring times

\[ X_{t \sim \bar{t}} | i_t \sim N(\mu_{t \sim \bar{t}}, \sigma^2_{t \sim \bar{t}}) \]

• **The views**: are statements (constraints) on the yet-to-be-defined distribution of the process:

\[ \nu_t : \left\{ \begin{array}{ll}
E_t^g \{ \nu_{\mu,t} X_{t \sim \bar{t}} \} & \equiv \mu_{\text{view};t} \\
C_t^g \{ \nu_{\sigma,t} X_{t \sim \bar{t}} \} & \equiv \sigma^2_{\text{view};t}.
\end{array} \right. \]

where \( \nu_{\mu,t} \) and \( \nu_{\sigma,t} \) are matrices that define arbitrary linear combinations of the process at the times for the views.

• **The posterior**: is the closest distribution to the prior that satisfies the views:

\[ \bar{f} \equiv \arg\min_{g \in \nu_t} \{ \mathcal{E} (g, f) \} \Rightarrow X_{t \sim \bar{t}} | i_t \sim N(\bar{\mu}_{t \sim \bar{t}}, \bar{\sigma}^2_{t \sim \bar{t}}) \]
Market model

\[ X_{t \sim \tilde{t}} | i_t \sim N(\bar{\mu}_{t \sim \tilde{t}}, \bar{\sigma}^2_{t \sim \tilde{t}}) \]

- For the expectation, we introduce the pseudo inverse matrix of \( \mathbf{v}_{\mu,t} \)
  \[ \mathbf{v}_{\mu,t}^+ \equiv \mathbf{\sigma}^2_{t \sim \tilde{t}} \mathbf{v}_{\mu,t}' \left( \mathbf{v}_{\mu,t} \mathbf{\sigma}^2_{t \sim \tilde{t}} \mathbf{v}_{\mu,t}' \right)^{-1} \]
  we define the two complementary projectors:
  \[ \mathbb{P}_{\mu,t} \equiv \left( \mathbb{1}_n(t-t+1) - \mathbf{v}_{\mu,t}^+ \mathbf{v}_{\mu,t} \right) \quad \mathbb{P}_{\mu,t}^\perp \equiv \mathbf{v}_{\mu,t}^+ \mathbf{v}_{\mu,t} \]
  Then
  \[ \bar{\mu}_{t \sim \tilde{t}} \equiv \mathbb{P}_{\mu,t} \mathbf{\mu}_{t \sim \tilde{t}} + \mathbb{P}_{\mu,t}^\perp \left( \mathbf{v}_{\mu,t}^+ \mathbf{\mu}_{view;t} \right) \]

- Similar, for the covariance we introduce the pseudo inverse of \( \mathbf{v}_{\sigma,t} \)
  \[ \mathbf{v}_{\sigma,t}^+ \equiv \mathbf{\sigma}^2_{t \sim \tilde{t}} \mathbf{v}_{\sigma,t}' \left( \mathbf{v}_{\sigma,t} \mathbf{\sigma}^2_{t \sim \tilde{t}} \mathbf{v}_{\sigma,t}' \right)^{-1} \]
  and the two complementary projectors:
  \[ \mathbb{P}_{\sigma,t} \equiv \mathbb{1}_n(t-t+1) - \mathbf{v}_{\sigma,t}^+ \mathbf{v}_{\sigma,t} \quad \mathbb{P}_{\sigma,t}^\perp \equiv \mathbf{v}_{\sigma,t}^+ \mathbf{v}_{\sigma,t} \]
  Then
  \[ \bar{\sigma}^2_{t \sim \tilde{t}} \equiv \mathbb{P}_{\sigma,t} \mathbf{\sigma}^2_{t \sim \tilde{t}} \mathbb{P}_{\sigma,t}' + \mathbb{P}_{\sigma,t}^\perp \left( \mathbf{v}_{\sigma,t}^+ \mathbf{\sigma}^2_{view;t} \left( \mathbf{v}_{\sigma,t}^+ \right)' \right) \left( \mathbb{P}_{\sigma,t}^\perp \right)' \]
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• As in Garleanu and Pedersen (2013), the satisfaction functional is an infinite sum of discounted trade-offs:

\[
\bar{S}_t^{(\gamma, \eta)} \equiv \sum_{s=t}^{\infty} e^{-\lambda(s-t)} \left[ \bar{E} \left\{ \Pi_{(s,s+1)} | \mathbf{i}_t \right\} - \frac{\gamma}{2} \bar{V} \left\{ \Pi_{(s,s+1)} | \mathbf{i}_t \right\} - \frac{\eta}{2} \bar{E} \left\{ MI_s | \mathbf{i}_t \right\} \right]
\]

where the market impact is a quadratic function of the exposure rebalancing

\[
MI_t = a^2 + \Delta b'_t c^2 \Delta b_t
\]

with \( c^2 \) a suitable positive definite matrix. Note the term \( a^2 \), which represents the average cost of maintaining constant exposures.
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Portfolio construction

Objective as function of exposures

- Given that the P&L is linear in the exposures $\Pi_{t+1} = b_t' \Delta X_{t+1}$, we need to solve for the optimal policy of exposures as functions of information

$$\{b^*_s = p^*_s (i_s) \}_{s \geq t}$$

where

$$\{p^*_s\}_{s \geq t} = \arg\max_{\{p_s\}_{s \geq t} \in \mathcal{C}} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} - \frac{\gamma}{2} p_s(I_s)' \omega \bar{C} v_s \{\Delta X_{s+1}\} \omega' p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)' c^2 \Delta p_s(I_s)] \right\}$$

- As in Garleanu and Pedersen (2013), the satisfaction functional is an infinite sum of discounted trade-offs:

$$\overline{S}_t^{(\gamma, \eta)} = \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [\mathbb{E}_s \{\Pi_{(s,s+1)} | i_t\} - \frac{\gamma}{2} \mathbb{V}_s \{\Pi_{(s,s+1)} | i_t\} - \frac{\eta}{2} \mathbb{E}_s \{MI_s | i_t\}]$$

where the market impact is a quadratic function of the exposure rebalancing

$$MI_t = a^2 + \Delta b_t' c^2 \Delta b_t$$

with $c^2$ a suitable positive definite matrix. Note the term $a^2$, which represents the average cost of maintaining constant exposures.
Given that the P&L is linear in the exposures $\Pi_{t+1} = b'_t \Delta X_{t+1}$, we need to solve for the optimal policy of exposures as functions of information

$$\{b_s^* = p_s^*(i_s)\}_{s \geq t}$$

where

$$\{p_s^*\}_{s \geq t} = \text{argmax}_{\{p_s\}_{s \geq t}} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} - \frac{\gamma}{2} p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} \omega' p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)' \mathbf{c}^2 \Delta p_s(I_s)] \right\}$$

Dynamic programming with a quadratic value function yields a recursive problem with time-dependent coefficients

$$v_{s+1}(b_s, x_{s+1}) = -\frac{1}{2} b'_s \psi_{bb,s} b_s + b'_s \psi_{bx,s} x_{s+1} + \frac{1}{2} x_{s+1}' \psi_{xx,s} x_{s+1} + \psi'_{b,s} b_s + \psi'_{x,s} x_{s+1} + \psi_{0,s}$$

$$\iff \psi_{s-1} = g_s(\psi_s)$$

The optimal policy of exposures then reads

$$b_s^* = (\gamma \omega \bar{\sigma}_s^2 \omega' + \eta \mathbf{c}^2 + e^{-\lambda} \psi_{bb,s})^{-1} \left[ \eta \mathbf{c}^2 b_{s-1} \right]$$

$$\begin{align*}
&\text{legacy exposures} \\
+ (\omega \beta_s + e^{-\lambda} \psi_{bx,s}(\beta_s + \mathbb{I}_{\bar{n}})) x_s + (\omega + e^{-\lambda} \psi_{bx,s}) \alpha_s + e^{-\lambda} \psi_{b,s} \end{align*}$$

$$\begin{align*}
&\text{current risk drivers} \\
&\text{(*) future views}
\end{align*}$$
Given that the P&L is linear in the exposures \( \Pi_{t+1} = b'_t \Delta X_{t+1} \), we need to solve for the optimal policy of exposures as functions of information

\[
\{ b_s^* = p_s^*(i_s) \}_{s \geq t}
\]

where

\[
\{ p_s^* \}_{s \geq t} = \arg \max_{\{ p_s \}_{s \geq t} \in \mathcal{C}} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [ p_s(I_s)' \omega \mathbb{E}_s \{ \Delta X_{s+1} \} - \frac{\gamma}{2} p_s(I_s)' \omega \mathbb{C}_s \omega p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)' c^2 \Delta p_s(I_s) ] \right\}
\]

With no market impact, we obtain a series of myopic one-period problems.

The optimal policy is a sequence of mean-variance optimizations based on the posterior distribution of the risk drivers process.

\[
b_s^* = \frac{1}{\gamma} \left( \omega \sigma_s^2 \omega' \right)^{-1} \omega \left( \mathbb{P}_{\mu, s} \right)_{s+1} \cdot \Delta \mu_{s \rightarrow t}^\text{LongTerm} - \frac{1}{\gamma} \left( \omega \sigma_s^2 \omega' \right)^{-1} \omega \left( \mathbb{P}_{\mu, s}^\perp \right)_{s+1} \cdot \Delta \mu_{s \rightarrow t}^\text{ViewMean} + \frac{1}{\gamma} \left( \omega \sigma_s^2 \omega' \right)^{-1} \omega \left( \mathbb{P}_{\mu, s}^\perp \right)_{s+1} \cdot \Delta \mu_{s \rightarrow t}^\text{ViewMean}
\]

\[
\begin{pmatrix}
\mathbb{I} - e^{-\theta} \\
\mathbb{I} - e^{-\theta} \\
\mathbb{I} - e^{-(\bar{t}-s)\theta}
\end{pmatrix}
\begin{pmatrix}
\mu - x_s
\end{pmatrix}
\]
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$P_{Prior}$

$P_{LongTerm}$

$P_{viewMean}$

One risk driver, one view

$P^*$

$t (years)$
Case studies

Two risk drivers (one investable), two views