Data Analytics and Technical Computing Workflow

Data Exploration
- Gain Insights
- Filter Data
- Build Intuition
- Hypothesize

Analytics Development
- Create prototype
- Machine Learning
- Optimization
- App Development

Analytics Integration
- Version Control
- Testing Code
- Validation
- Deploy & Share

HDFS

SERVER

Desktop

Web Application

MATLAB Production Server(s)

Web Server(s)
Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers
- Exploring Optimization Solvers in MATLAB
- Key Takeaways
Numerical Optimization Workflow
Numerical Optimization Workflow
Numerical Optimization Workflow

Initial Design Variables → System → Optimal Design
Numerical Optimization Workflow

Initial Design Variables → System → Objectives met?

- No
- Yes → Optimal Design
Numerical Optimization Workflow

- Initial Design Variables
  - Modify Design Variables
  - System
  - Objectives met?
    - No
    - Yes: Optimal Design
What Is Modeling?

- Modeling: process of turning a problem into a mathematical statement
- Problem can be words, pictures, etc.
- Modeling includes deciding what to keep, and naming variables
- Modeling is an art, not a science
Steps in Modeling

1. Get an overall idea of the problem
2. What is the goal? What are you trying to achieve?
3. Identify variables
4. Identify constraints
5. Identify the inputs and outputs you can control
6. Specify all quantities mathematically
7. Check the model for completeness and correctness
1. Overall Idea

Model adapted from “Optimization of Chemical Processes” by Edgar and Himmelblau, McGraw-Hill, 1988
1. Overall Idea

Costs

- Fuel: $0.002614/lb high-pressure steam
- Electricity purchased: $0.0239/kWh
- Penalty for purchasing less than 12,000 kW: $0.009825/kWh

Demands

- Medium-pressure steam: 271,536 lb/h
- Low-pressure steam: 100,623 lb/h
- Electric power: 24,550 kW
2. What Is the Goal?

Objective

- Run the plant at minimal cost
- In other words, satisfy the constraints using the least amount of money
3. Identify Variables

Turbine 1 Model

- Identify Variables
  - \(I_1\) = inlet flow rate
  - \(C\) = condensate flow rate
  - \(LE_1\) = low-pressure steam flow rate
  - \(HE_1\) = medium-pressure steam flow rate
  - \(P_1\) = power generated by Turbine 1

Units
- Flow rate: lb/h
- Power: kW
4. Identify Constraints
Turbine 1 Model

- $2500 \leq P_1 \leq 6250$ (capacity)
- $I_1 \leq 192,000$ (max inlet flow)
- $C \leq 62,000$ (max condensate)
- $I_1 - HE_1 \leq 132,000$ (max internal)
- $I_1 = LE_1 + HE_1 + C$ (mass conservation)
- $1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + 1251.4 \cdot LE_1$
  $+ 192 \cdot C + 3413 \cdot P_1$ (energy cons.)

Flow rate lb/h, Power kW, Energy BTU
3&4. Identify Variables/Constraints
Turbine 2 Model

- $3000 \leq P_2 \leq 9000$
- $I_2 \leq 244,000$ (max inlet flow)
- $LE_2 \leq 142,000$ (max low-pressure flow)
- $I_2 = LE_2 + HE_2$ (mass conservation)
- $1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + 1251.4 \cdot LE_2$
  $+ 3413 \cdot P_2$ (energy conservation)

Flow rate lb/h, Power kW, Energy BTU
4. Identify Constraints

Material Balance and Demands

- **HPS =** high-pressure steam (MPS, LPS)
- **BFi =** flow through valve i

- **HPS = I1 + I2 + BF1**
- **HPS = C + MPS + LPS**
- **MPS = HE1 + HE2 + BF1 − BF2**

- **LPS = LE1 + LE2 + BF2**
- **MPS ≥ 271,536**
- **LPS ≥ 100,623**
4. Identify Constraints

Electric Power Expressions

- $PP = \text{purchased power}$
- $EP = \text{excess power (unpurchased)}$

- $P1 + P2 + PP \geq 24,550$
- $EP + PP \geq 12,000$
5 – 7 Model Summary – Collected Equations

- $2500 \leq P1 \leq 6250$
- $I1 \leq 192,000$
- $C \leq 62,000$
- $I1 - HE1 \leq 132,000$
- $I1 = LE1 + HE1 + C$
- $1359.8 \cdot I1 = 1267.8 \cdot HE1$
  + $1251.4 \cdot LE1 + 192 \cdot C + 3413 \cdot P1$
- $3000 \leq P2 \leq 9000$
- $I2 \leq 244,000$
- $LE2 \leq 142,000$
- $I2 = LE2 + HE2$
- $1359.8 \cdot I2 = 1267.8 \cdot HE2$
  + $1251.4 \cdot LE2 + 3413 \cdot P2$

- $HPS = I1 + I2 + BF1$
- $HPS = C + MPS + LPS$
- $LPS = LE1 + LE2 + BF2$
- $MPS = HE1 + HE2 + BF1 - BF2$
- $P1 + P2 + PP \geq 24,550$
- $EP + PP \geq 12,000$
- $MPS \geq 271,536$
- $LPS \geq 100,623$
- All variables positive

- Cost = $0.002614 \cdot HPS + 0.0239 \cdot PP + 0.009825 \cdot EP$
# Find Solver and Syntax

## Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Least Squares</th>
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</table>
Find Solver and Syntax

- linprog:

\[ \min_x f^T x \text{ such that } \begin{cases} \ A \cdot x \leq b, \\ \ A_{eq} \cdot x = b_{eq}, \\ \ lb \leq x \leq ub. \end{cases} \]

- Syntax:

\[ [x \ fval] = \text{linprog}(f, A, b, A_{eq}, b_{eq}, lb, ub) \]

- Syntax implies linear inequalities, linear equalities, and bounds should be separated
Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector \((x)\)
3. Write vectors for lower and upper bounds \((lb, ub)\)
4. Write matrix and vector of inequalities \((A, b)\)
5. Write matrix and vector of equalities \((Aeq, beq)\)
6. Write nonlinear constraint function
7. Write the objective (function, or vector \(f\))
8. Call the solver
Step 1: Separate Bounds — Inequalities Involving One Variable

- $2500 \leq P_1 \leq 6250$
- $I_1 \leq 192,000$
- $C \leq 62,000$
- $3000 \leq P_2 \leq 9000$
- $I_2 \leq 244,000$
- $LE_2 \leq 142,000$
- $MPS \geq 271,536$
- $LPS \geq 100,623$
- All variables are positive
Step 1: Separate Linear Constraints (In)Equalities Involving Two+ Variables

- \( I_1 - HE_1 \leq 132,000 \)
- \( I_2 = LE_2 + HE_2 \)
- \( LPS = LE_1 + LE_2 + BF_2 \)
- \( EP + PP \geq 12,000 \)
- \( HPS = I_1 + I_2 + BF_1 \)
- \( HPS = C + MPS + LPS \)
- \( I_1 = LE_1 + HE_1 + C \)
- \( MPS = HE_1 + HE_2 + BF_1 - BF_2 \)

- \( 1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + 1251.4 \cdot LE_1 + 192 \cdot C + 3413 \cdot P_1 \)
- \( 1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + 1251.4 \cdot LE_2 + 3413 \cdot P_2 \)
- \( P_1 + P_2 + PP \geq 24,550 \)
Step 2: Combine Variables into One Vector

\[ X_k = X(k) \]

- \( X_1 = I1 \)
- \( X_2 = I2 \)
- \( X_3 = HE1 \)
- \( X_4 = HE2 \)
- \( X_5 = LE1 \)
- \( X_6 = LE2 \)
- \( X_7 = C \)
- \( X_8 = BF1 \)
- \( X_9 = BF2 \)
- \( X_{10} = HPS \)
- \( X_{11} = MPS \)
- \( X_{12} = LPS \)
- \( X_{13} = P1 \)
- \( X_{14} = P2 \)
- \( X_{15} = PP \)
- \( X_{16} = EP \)
Step 3: Write the Lower Bounds Vector

- All variables positive
- $2500 \leq P1 \ (X_{13})$
- $3000 \leq P2 \ (X_{14})$
- $271,536 \leq MPS \ (X_{11})$
- $100,623 \leq LPS \ (X_{12})$

```matlab
lb = zeros(16,1);
lb(13) = 2500;
lb(14) = 3000;
lb(11) = 271536;
lb(12) = 100623;
```
Step 3: Write the Upper Bounds Vector

- $P_1 \leq 6250 \ (X_{13})$
  - $ub = \inf(16,1)$;
  - $ub(13) = 6250$;

- $I_1 \leq 192,000 \ (X_1)$
  - $ub(1) = 192000$;

- $C \leq 62,000 \ (X_7)$
  - $ub(7) = 62000$;

- $P_2 \leq 9000 \ (X_{14})$
  - $ub(14) = 9000$;

- $I_2 \leq 244,000 \ (X_2)$
  - $ub(2) = 244000$;

- $LE_2 \leq 142,000 \ (X_6)$
  - $ub(6) = 142000$;
Step 4: Write Linear Inequality Matrix and Vector

$$A \cdot x \leq b$$

- $I_1 - HE_1 \leq 132,000$
- $EP + PP \geq 12,000$
- $P_1 + P_2 + PP \geq 24,550$
- Put into “less than” form:
  - $I_1 - HE_1 \leq 132,000$;
  - $-EP - PP \leq -12,000$;
  - $-P_1 - P_2 - PP \leq -24,550$;

$$X_1 - X_3 \leq 132,000$$
$$-X_{16} - X_{15} \leq -12,000$$
$$-X_{13} - X_{14} - X_{15} \leq -24,550$$
Step 4: Write Linear Inequality
Matrix and Vector

\[ A \cdot x \leq b \]

- \( X_1 - X_3 \leq 132,000 \)
- \(-X_{16} - X_{15} \leq -12,000 \)
- \(-X_{13} - X_{14} - X_{15} \leq -24,550 \)

Matrix A and vector b for \( AX \leq b \)
- There are three inequalities in 16 variables
- Therefore A should have 3 rows and 16 columns
- b should be a column vector with 3 entries
Step 4: Write Linear Inequality Matrix and Vector

\[ A \cdot x \leq b \]

- \( X_1 - X_3 \leq 132,000 \)
- \(-X_{16} - X_{15} \leq -12,000 \)
- \(-X_{13} - X_{14} - X_{15} \leq -24,550 \)

Matrix A and vector b for \( AX \leq b \)

\[
A = \text{zeros}(3,16);
\]

\[
A(1,1) = 1; \quad A(1,3) = -1; \quad b(1) = 132000;
\]

\[
A(2,15:16) = -1; \quad b(2) = -12000;
\]

\[
A(3,13:15) = -1; \quad b(3) = -24550;
\]
Step 5: Write Linear Equality Matrix and Vector

- $I_2 = LE_2 + HE_2$
- $LPS = LE_1 + LE_2 + BF_2$
- $HPS = I_1 + I_2 + BF_1$
- $HPS = C + MPS + LPS$
- $I_1 = LE_1 + HE_1 + C$
- $MPS = HE_1 + HE_2 + BF_1 - BF_2$
- $1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + 1251.4 \cdot LE_1 + 192 \cdot C + 3413 \cdot P_1$
- $1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + 1251.4 \cdot LE_2 + 3413 \cdot P_2$
Step 5: Write Linear Equality Matrix and Vector

- LE2 + HE2 – I2 = 0
- LE1 + LE2 + BF2 – LPS = 0
- I1 + I2 + BF1 – HPS = 0
- C + MPS + LPS – HPS = 0
- LE1 + HE1 + C – I1 = 0
- HE1 + HE2 + BF1 – BF2 – MPS = 0
- 1267.8·HE1 + 1251.4·LE1 + 192·C + 3413·P1 – 1359.8·I1 = 0
- 1267.8·HE2 + 1251.4·LE2 + 3413·P2 – 1359.8·I2 = 0

- X6 + X4 – X2 = 0
- X5 + X6 + X9 – X12 = 0
- X1 + X2 + X8 – X10 = 0
- X7 + X11 + X12 – X10 = 0
- X5 + X3 + X7 – X1 = 0
- X3 + X4 + X8 – X9 – X11 = 0
- 1267.8·X3 + 1251.4·X5 + 192·X7 + 3413·X13 – 1359.8·X1 = 0
- 1267.8·X4 + 1251.4·X6 + 3413·X14 – 1359.8·X2 = 0
Step 5: Write Linear Equality Matrix, Vector

Equation: $Aeq \, X = \, beq$

8 equations in 16 variables: $Aeq$ is 8 by 16, $beq$ is 8 by 1

$Aeq = \text{zeros}(8,16); \, beq = \text{zeros}(8,1);$
Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector \((x)\)
3. Write vectors for lower and upper bounds \((lb, ub)\)
4. Write matrix and vector of inequalities \((A, b)\)
5. Write matrix and vector of equalities \((Aeq, beq)\)
6. Write nonlinear constraint function
7. **Write the objective** (function, or vector \(f\))
8. Call the solver
Step 7: Write Objective Function Vector

- Cost = 0.002614 \cdot HPS + 0.0239 \cdot PP + 0.009825 \cdot EP

  \[ X_{10} \quad X_{15} \quad X_{16} \]

- Equation: minimize cost = \( f^T X = f(1)X_1 + \ldots + f(16)X_{16} \)

\[
\begin{align*}
f &= \text{zeros}(16,1); \\
f(10) &= 0.002614; \\
f(15) &= 0.0239; \\
f(16) &= 0.009825;
\end{align*}
\]
Step 8: Call Solver and Obtain Solution

\[ [x \ fval] = \text{linprog}(f,A,b,Aeq,beq,lb,ub) \]

Optimization terminated.

\[
\begin{align*}
x &= \\
1.0e+005 & \times \\
1.3633 & \quad 2.7154 \\
2.4400 & \quad 1.0062 \\
1.2816 & \quad 0.0625 \\
1.4338 & \quad 0.1124 \\
0.0000 & \quad 0.0706 \\
1.0062 & \quad 0.0076 \\
0.0817 & \\
0.0000 & \\
0.0000 & \\
3.8033 &
\end{align*}
\]

fval = 1.2703e+003

Cost for operating plant: $1270/hour
Examine the Solution

- **Cost breakdown:**
  - Fuel HPS \( X(10) \times f(10) = 994.18 \) \$/hr
  - Purchased Power PP \( X(15) \times f(15) = 268.62 \) \$/hr
  - Penalty EP \( X(16) \times f(16) = 7.47 \) \$/hr

- **Operating conditions**
  - Turbine 1 I1 \( X(1) = 136,329 \) lbs/hr (max 192,000)
  - Turbine 2 I2 \( X(2) = 244,000 \) lbs/hr (max 244,000)
  - Valves BF1 BF2 \( X(8) = X(9) = 0 \)
  - Condensate C \( X7 = 8170 \) lbs/hr (max 62,000)
Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers

- Exploring Optimization Solvers in MATLAB

- Key Takeaways
Problem 1: Model Gravity as a Function of Altitude

Problem Statement:
A simple model for the acceleration due to gravity $g$ as a function of altitude $h$ is given by $g(h) = ah + b$, where $a$ and $b$ are unknown coefficients.

Given the following experimental data for $g$ at various values of $h$
$g=[9.8100 9.7487 9.6879 9.6278 9.5682]$ and $h=[0 20 40 60 80]$

Find the coefficients $a$ and $b$ that best fits the model to the experimental data.

Problem Identification:
- This is unconstrained least squares curve fitting problem

Solution:
- Can be solved using “\” in basic MATLAB
## Find Solver and Syntax

<table>
<thead>
<tr>
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<th>Linear</th>
<th>Quadratic</th>
<th>Objective Type</th>
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<td>n/a ( f = \text{const, or min } = -\infty )</td>
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<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
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</tbody>
</table>
Problem 2: Find Minima of Y = F(X), X has bounds

**Problem Statement:**
Find the minima of the function.
\[ F(x) = x \cdot \sin(x) + x \cdot \cos(2 \cdot x) \]

Given 0 < x < 10

**Problem Identification:**
- This is a bounded/ constrained one dimensional optimization problem

**Solution:**
- Can be solved using "fminbnd" in basic MATLAB
## Find Solver and Syntax

<table>
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Problem 3: Compute Volumetric Efficiency of an Engine

Problem Statement:
Find the optimal values for the manifold pressure ratio and the engine revolutions in RPM, to attain the maximum volumetric efficiency.

<table>
<thead>
<tr>
<th>Engine Speed (RPM)</th>
<th>Manifold Pressure Ratio</th>
<th>Volumetric Efficiency</th>
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<tbody>
<tr>
<td>3500</td>
<td>0.5</td>
<td>0.896</td>
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<tr>
<td>3500</td>
<td>0.6</td>
<td>0.915</td>
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<tr>
<td>3500</td>
<td>0.7</td>
<td>0.929</td>
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<tr>
<td>3500</td>
<td>0.8</td>
<td>0.887</td>
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<tr>
<td>3500</td>
<td>0.75</td>
<td>0.937</td>
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<tr>
<td>3000</td>
<td>0.75</td>
<td>0.862</td>
</tr>
<tr>
<td>4000</td>
<td>0.75</td>
<td>0.913</td>
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Problem Identification:
- This is a non-smooth unconstrained multivariable optimization problem

Solution:
- Can be solved using “fminsearch” in Optimization Toolbox

Optimal Params

<table>
<thead>
<tr>
<th>Engine Speed (RPM)</th>
<th>Manifold Pressure Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4300</td>
<td>0.67</td>
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</tbody>
</table>

Best Value

<table>
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<tr>
<th>Volumetric Efficiency</th>
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<td>0.973</td>
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# Find Solver and Syntax

## Solvers by Objective and Constraint

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</table>
Problem 4: Hydroelectric Dam Optimization
Equations

Control Volume for Reservoir:

(1) \( \text{Storage}(t) = \text{Storage}(t - 1) + \ldots \)

\( \ldots \Delta t \ast [\text{inFlow}(t - 1) - \text{spillFlow}(t - 1) - \text{turbineFlow}(t - 1)] \)

Equation for Electricity:

(2) \( \text{Electricity}(t) = \text{TurbineFlow}(t - 1) \ast \left[ \frac{1}{2} k_1 (\text{Storage}(t) + \text{Storage}(t - 1)) + k_2 \right] \)
Decision Variables

- Turbine Flow at each time step
- Spill Flow at each time step

\[ x = \begin{pmatrix} TF_0 \\ TF_1 \\ \vdots \\ TF_n \\ SF_0 \\ SF_1 \\ \vdots \\ SF_n \end{pmatrix} \]
**Constraints**

1. $0 \leq \text{turbineFlow} \leq 25000$ CFS
2. $0 \leq \text{spillFlow}$
3. $\text{projectFlow} \geq 500$ CFS
4. $\text{projectFlow}(t) - \text{projectFlow}(t - 1) \leq 500$ CFS
5. $50000 \leq \text{storage} \leq 100000$ AF
6. $\text{storage}_{t=\text{end}} = \text{storage}_{t=0}$
Objective Formulation

(1) Quadratic function:

\[ f = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{4} x_1 x_2 + \frac{1}{4} x_1 - \frac{1}{2} x_2 \]
Objective Formulation

(1) **Quadratic function:**

\[
f = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{4} x_1 x_2 + \frac{1}{4} x_1 - \frac{1}{2} x_2
\]

(2) **Gradient:**

\[
\frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4} x_2 + \frac{1}{4}
\]

\[
\frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4} x_1 - \frac{1}{2}
\]
Objective Formulation

(1) **Quadratic function:**

\[ f = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{4} x_1 x_2 + \frac{1}{4} x_1 - \frac{1}{2} x_2 \]

(2) **Gradient:**

\[ \frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4} x_2 + \frac{1}{4} \]

\[ \frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4} x_1 - \frac{1}{2} \]

(3) **Plug in zeros for x:**

\[ c_{x_1} = \frac{1}{4} \]

\[ c_{x_2} = -\frac{1}{2} \]
## Find Solver and Syntax

### Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Objective Type</th>
<th>Objective Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>n/a (f = \text{const, or min} = -\infty)</td>
<td>quadprog, Information</td>
<td>\text{lsqlinfit, lsqnonlin, Information}</td>
<td>\text{fminsearch, fminunc, Information}</td>
</tr>
<tr>
<td>Bound</td>
<td>linprog, Information</td>
<td>quadprog, Information</td>
<td>lsqlin, Information</td>
<td>fmincon, fseminf, Information</td>
</tr>
<tr>
<td>Linear</td>
<td>linprog, Information</td>
<td>quadprog, Information</td>
<td>lsqlin, Information</td>
<td>fmincon, fseminf, Information</td>
</tr>
<tr>
<td>General smooth</td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>fmincon, fseminf, Information</td>
</tr>
<tr>
<td>Discrete</td>
<td>intlinprog, Information</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>
## Demo: Scaling Up to a Larger Problem

### Time to Run Optimization (seconds)

<table>
<thead>
<tr>
<th>Decision Variables</th>
<th>FMINCON (no gradient)</th>
<th>FMINCON (with Hessian)</th>
<th>QUADPROG interior-point</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>16</td>
<td>0.46</td>
<td>0.04</td>
</tr>
<tr>
<td>200</td>
<td>238</td>
<td>1.24</td>
<td>0.16</td>
</tr>
<tr>
<td>400</td>
<td>965</td>
<td>3.17</td>
<td>0.77</td>
</tr>
<tr>
<td>800</td>
<td>6083</td>
<td>43</td>
<td>5</td>
</tr>
<tr>
<td>1600</td>
<td>-</td>
<td>*</td>
<td>30</td>
</tr>
<tr>
<td>3200</td>
<td>-</td>
<td>*</td>
<td>190</td>
</tr>
<tr>
<td>6400</td>
<td>-</td>
<td>*</td>
<td>1641</td>
</tr>
</tbody>
</table>

*Gradient/Hessian generation took several hours*
Optimization toolboxes support different problem types

<table>
<thead>
<tr>
<th>Objective Types</th>
<th>Constraint Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>Unconstrained</td>
</tr>
<tr>
<td>Quadratic</td>
<td>Bound</td>
</tr>
<tr>
<td>Sum-of-squares (Least Squares)</td>
<td>Linear</td>
</tr>
<tr>
<td>Smooth nonlinear</td>
<td>General Smooth</td>
</tr>
<tr>
<td>Nonsmooth</td>
<td>Discrete (integer)</td>
</tr>
</tbody>
</table>

**Linear Objective**

\[-3x_1 + 5x_2 < 15\]

\[3x_1 - 2x_2 < 6\]

**Quadratic Objective**

\[18x_1 + x_2 < 19.5\]

\[3x_1 - 2x_2 < 4\]

\[-2x_1 - 3x_2 < 4\]
Optimization problems

Linear
- LINPROG

Mixed Integer
- INTLINPROG

Quadratic
- QUADPROG

Nonlinear
- FMINCON
- MULTISTART
- GLOBALSEARCH
- PATTERNSEARCH

Least Squares
- LSQLIN
- LSQNONNEG
- LSQCURVEFIT
- LSQNONLIN

Nonsmooth or Noisy
- PATTERNSEARCH
- GA
- SIMULANNEALBND

Multiobjective
- GAMULTIOBJ
Key takeaways

1. Solve Wide Variety of Problems
   - Linear, quadratic, nonlinear, least squares (Optimization Toolbox)
   - Nonlinear, nonsmooth, stochastic, noisy (Global Optimization Toolbox)

2. Set up, Run, and Monitor Optimizations
   - Optimization App
   - MATLAB functions
   - Plot functions, command line display

3. MATLAB Environment
   - Integrated Numerics, Graphics, Symbolic Math
   - Parallel Computing