Numerical Optimization Using MATLAB

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Data Analytics and Technical Computing Workflow

HDFS

SERVER

Desktop

Web Application

Data Exploration
- Gain Insights
- Filter Data
- Build Intuition
- Hypothesize

Analytics Development
- Create prototype
- Machine Learning
- **Optimization**
- App Development

Analytics Integration
- Version Control
- Testing Code
- Validation
- Deploy & Share

SERVER

MATLAB
Production
Server(s)

Web
Server(s)
Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers
- Exploring Optimization Solvers in MATLAB
- Key Takeaways
Numerical Optimization Workflow
Numerical Optimization Workflow

Initial Design Variables → System
Numerical Optimization Workflow

Initial Design Variables → System → Optimal Design
Numerical Optimization Workflow
Numerical Optimization Workflow

1. Initial Design Variables
2. System
3. Modify Design Variables
4. Objectives met?
   - Yes: Optimal Design
   - No: Repeat from Initial Design Variables
What Is Modeling?

- Modeling: process of turning a problem into a mathematical statement
- Problem can be words, pictures, etc.
- Modeling includes deciding what to keep, and naming variables
- Modeling is an art, not a science
Steps in Modeling

1. Get an overall idea of the problem
2. What is the goal? What are you trying to achieve?
3. Identify variables
4. Identify constraints
5. Identify the inputs and outputs you can control
6. Specify all quantities mathematically
7. Check the model for completeness and correctness
1. Overall Idea

Model adapted from “Optimization of Chemical Processes” by Edgar and Himmelblau, McGraw-Hill, 1988
1. Overall Idea

Costs

- Fuel: $0.002614/lb high-pressure steam
- Electricity purchased: $0.0239/kWh
- Penalty for purchasing less than 12,000 kW: $0.009825/kWh

Demands

- Medium-pressure steam: 271,536 lb/h
- Low-pressure steam: 100,623 lb/h
- Electric power: 24,550 kW
2. What Is the Goal?

Objective

- Run the plant at minimal cost

- In other words, satisfy the constraints using the least amount of money
3. Identify Variables

Turbine 1 Model

- Identify Variables
  - $I_1 = \text{inlet flow rate}$
  - $C = \text{condensate flow rate}$
  - $LE_1 = \text{low-pressure steam flow rate}$
  - $HE_1 = \text{medium-pressure steam flow rate}$
  - $P_1 = \text{power generated by Turbine 1}$

Units
Flow rate: lb/h
Power: kW
4. Identify Constraints
Turbine 1 Model

- $2500 \leq P_1 \leq 6250$ (capacity)
- $I_1 \leq 192,000$ (max inlet flow)
- $C \leq 62,000$ (max condensate)
- $I_1 - HE_1 \leq 132,000$ (max internal)
- $I_1 = LE_1 + HE_1 + C$ (mass conservation)
- $1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + 1251.4 \cdot LE_1 + 192 \cdot C + 3413 \cdot P_1$ (energy cons.)

Flow rate lb/h, Power kW, Energy BTU
3&4. Identify Variables/Constraints
Turbine 2 Model

- $3000 \leq P_2 \leq 9000$
- $I_2 \leq 244,000$ (max inlet flow)
- $LE_2 \leq 142,000$ (max low-pressure flow)
- $I_2 = LE_2 + HE_2$ (mass conservation)
- $1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + 1251.4 \cdot LE_2 + 3413 \cdot P_2$ (energy conservation)

Flow rate $\text{lb/h}$, Power $\text{kW}$, Energy $\text{BTU}$
4. Identify Constraints

Material Balance and Demands

- HPS = high-pressure steam (MPS, LPS)
- BF\textsubscript{i} = flow through valve \textit{i}

- HPS = I\textsubscript{1} + I\textsubscript{2} + BF\textsubscript{1}
- HPS = C + MPS + LPS
- MPS = HE\textsubscript{1} + HE\textsubscript{2} + BF\textsubscript{1} – BF\textsubscript{2}

- LPS = LE\textsubscript{1} + LE\textsubscript{2} + BF\textsubscript{2}
- MPS \geq 271,536
- LPS \geq 100,623
4. Identify Constraints
Electric Power Expressions

- PP = purchased power
- EP = excess power (unpurchased)

- P1 + P2 + PP ≥ 24,550
- EP + PP ≥ 12,000
5 – 7 Model Summary – Collected Equations

- **Turbine 1**
  - \(2500 \leq P_1 \leq 6250\)
  - \(I_1 \leq 192,000\)
  - \(C \leq 62,000\)
  - \(I_1 - HE_1 \leq 132,000\)
  - \(I_1 = LE_1 + HE_1 + C\)
  - \(1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + 1251.4 \cdot LE_1 + 192 \cdot C + 3413 \cdot P_1\)
  - \(3000 \leq P_2 \leq 9000\)
  - \(I_2 \leq 244,000\)
  - \(LE_2 \leq 142,000\)
  - \(I_2 = LE_2 + HE_2\)
  - \(1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + 1251.4 \cdot LE_2 + 3413 \cdot P_2\)

- **Turbine 2**

- **HPS** = \(I_1 + I_2 + BF_1\)
- **HPS** = \(C + MPS + LPS\)
- **LPS** = \(LE_1 + LE_2 + BF_2\)
- **MPS** = \(HE_1 + HE_2 + BF_1 - BF_2\)
- \(P_1 + P_2 + PP \geq 24,550\)
- \(EP + PP \geq 12,000\)
- \(MPS \geq 271,536\)
- \(LPS \geq 100,623\)
- All variables positive

- **Cost** = \(0.002614 \cdot HPS + 0.0239 \cdot PP + 0.009825 \cdot EP\)
# Find Solver and Syntax

## Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Objective Type</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Least Squares</th>
<th>Smooth nonlinear</th>
<th>Nonsmooth</th>
</tr>
</thead>
</table>
| None            | n/a (f = const, or min = −∞) | quadprog, Information | lsqcurvefit, 
lsqnonlin, Information | fminsearch, fminunc, Information | fminsearch,* |               |
| Bound           | linprog, Information   | quadprog, Information | lsqcurvefit, 
lsqnonlin, Information | fminbnd, fmincon, fseminf, Information | fminbnd,*    |               |
| Linear          | linprog, Information   | quadprog, Information | lsqlin, Information | fmincon, fseminf, Information | *              |               |
| General smooth  | fmincon, Information   | fmincon, Information | fmincon, Information | fmincon, fseminf, Information | *              |               |
| Discrete        | intlinprog, Information | *               | *               | *               | *                |               |
Find Solver and Syntax

- **linprog**: \[
\min f^T x \text{ such that } \begin{cases} \mathbf{A} \cdot x \leq \mathbf{b}, \\
\mathbf{A}_{eq} \cdot x = \mathbf{b}_{eq}, \\
\mathbf{lb} \leq x \leq \mathbf{ub}. \end{cases}
\]

- **Syntax**: 
  \[
  [x \ fval] = \text{linprog}(f, \mathbf{A}, \mathbf{b}, \mathbf{A}_{eq}, \mathbf{b}_{eq}, \mathbf{lb}, \mathbf{ub})
  \]

- Syntax implies linear inequalities, linear equalities, and bounds should be separated
Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector \((x)\)
3. Write vectors for lower and upper bounds \((lb, ub)\)
4. Write matrix and vector of inequalities \((A, b)\)
5. Write matrix and vector of equalities \((Aeq, beq)\)
6. Write nonlinear constraint function
7. Write the objective (function, or vector \(f\))
8. Call the solver
Step 1: Separate Bounds — Inequalities Involving One Variable

- $2500 \leq P1 \leq 6250$
- $I1 \leq 192,000$
- $C \leq 62,000$
- $3000 \leq P2 \leq 9000$
- $I2 \leq 244,000$
- $LE2 \leq 142,000$
- $MPS \geq 271,536$
- $LPS \geq 100,623$
- All variables are positive
Step 1: Separate Linear Constraints (In)Equalities Involving Two+ Variables

- \( I_1 - HE_1 \leq 132,000 \)
- \( I_2 = LE_2 + HE_2 \)
- \( LPS = LE_1 + LE_2 + BF_2 \)
- \( EP + PP \geq 12,000 \)
- \( HPS = I_1 + I_2 + BF_1 \)
- \( HPS = C + MPS + LPS \)
- \( I_1 = LE_1 + HE_1 + C \)
- \( MPS = HE_1 + HE_2 + BF_1 - BF_2 \)

\[ 1251.4 \cdot LE_1 + 192 \cdot C + 3413 \cdot P_1 \]

- \[ 1359.8 \cdot I_2 = 1267.8 \cdot HE_2 + \]
- \[ 1251.4 \cdot LE_2 + 3413 \cdot P_2 \]
- \( P_1 + P_2 + PP \geq 24,550 \)

\[ 1359.8 \cdot I_1 = 1267.8 \cdot HE_1 + \]
Step 2: Combine Variables into One Vector

\[ X_k = X(k) \]

- \( X_1 = I1 \)
- \( X_2 = I2 \)
- \( X_3 = HE1 \)
- \( X_4 = HE2 \)
- \( X_5 = LE1 \)
- \( X_6 = LE2 \)
- \( X_7 = C \)
- \( X_8 = BF1 \)
- \( X_9 = BF2 \)
- \( X_{10} = HPS \)
- \( X_{11} = MPS \)
- \( X_{12} = LPS \)
- \( X_{13} = P1 \)
- \( X_{14} = P2 \)
- \( X_{15} = PP \)
- \( X_{16} = EP \)
Step 3: Write the Lower Bounds Vector

- All variables positive
- \(2500 \leq P1 \, (X_{13})\)
- \(3000 \leq P2 \, (X_{14})\)
- \(271,536 \leq MPS \, (X_{11})\)
- \(100,623 \leq LPS \, (X_{12})\)

\[
\text{lb} = \text{zeros}(16,1);
\]

\[
\text{lb}(13) = 2500; \quad \text{lb}(14) = 3000; \quad \text{lb}(11) = 271536; \quad \text{lb}(12) = 100623;
\]
Step 3: Write the Upper Bounds Vector

- \( P_1 \leq 6250 \) \((X_{13})\)
- \( I_1 \leq 192,000 \) \((X_1)\)
- \( C \leq 62,000 \) \((X_7)\)
- \( P_2 \leq 9000 \) \((X_{14})\)
- \( I_2 \leq 244,000 \) \((X_2)\)
- \( LE_2 \leq 142,000 \) \((X_6)\)

\[ ub = \text{Inf}(16,1); \]
\[ ub(13) = 6250; \]
\[ ub(1) = 192000; \]
\[ ub(7) = 62000; \]
\[ ub(14) = 9000; \]
\[ ub(2) = 244000; \]
\[ ub(6) = 142000; \]
Step 4: Write Linear Inequality Matrix and Vector

\[ A \cdot x \leq b \]

- \( I_1 - HE1 \leq 132,000 \)
- \( EP + PP \geq 12,000 \)
- \( P1 + P2 + PP \geq 24,550 \)
- Put into “less than” form:
  - \( I_1 - HE1 \leq 132,000; \)
  - \( -EP - PP \leq -12,000; \)
  - \( -P1 - P2 - PP \leq -24,550; \)

\[ X_1 - X_3 \leq 132,000 \]

\[ -X_{16} - X_{15} \leq -12,000 \]

\[ -X_{13} - X_{14} - X_{15} \leq -24,550 \]
Step 4: Write Linear Inequality Matrix and Vector

\[ A \cdot x \leq b \]

- \( X_1 - X_3 \leq 132,000 \)
- \(-X_{16} - X_{15} \leq -12,000 \)
- \(-X_{13} - X_{14} - X_{15} \leq -24,550 \)

Matrix A and vector b for \( AX \leq b \)

There are three inequalities in 16 variables

Therefore A should have 3 rows and 16 columns

b should be a column vector with 3 entries
Step 4: Write Linear Inequality
Matrix and Vector

\[ A \cdot x \leq b \]

- \( X_1 - X_3 \leq 132,000 \)
- \(-X_{16} - X_{15} \leq -12,000 \)
- \(-X_{13} - X_{14} - X_{15} \leq -24,550 \)
- Matrix A and vector b for \( AX \leq b \)

\[
A = \text{zeros}(3, 16); \\
A(1, 1) = 1; A(1, 3) = -1; b(1) = 132000; \\
A(2, 15:16) = -1; b(2) = -12000; \\
A(3, 13:15) = -1; b(3) = -24550;
\]
Step 5: Write Linear Equality Matrix and Vector

- \( I_2 = L\!E_2 + H\!E_2 \)
- \( LPS = L\!E_1 + L\!E_2 + B\!F_2 \)
- \( HPS = I_1 + I_2 + B\!F_1 \)
- \( HPS = C + M\!P\!S + LPS \)
- \( I_1 = L\!E_1 + H\!E_1 + C \)
- \( M\!P\!S = H\!E_1 + H\!E_2 + B\!F_1 - B\!F_2 \)
- \( 1359.8 \cdot I_1 = 1267.8 \cdot H\!E_1 + 1251.4 \cdot L\!E_1 + 192 \cdot C + 3413 \cdot P_1 \)
- \( 1359.8 \cdot I_2 = 1267.8 \cdot H\!E_2 + 1251.4 \cdot L\!E_2 + 3413 \cdot P_2 \)
Step 5: Write Linear Equality Matrix and Vector

- \( \text{LE2} + \text{HE2} - I2 = 0 \)
- \( \text{LE1} + \text{LE2} + \text{BF2} - \text{LPS} = 0 \)
- \( I1 + I2 + \text{BF1} - \text{HPS} = 0 \)
- \( C + \text{MPS} + \text{LPS} - \text{HPS} = 0 \)
- \( \text{LE1} + \text{HE1} + C - I1 = 0 \)
- \( \text{HE1} + \text{HE2} + \text{BF1} - \text{BF2} - \text{MPS} = 0 \)
- \( 1267.8 \cdot \text{HE1} + 1251.4 \cdot \text{LE1} + 192 \cdot C + 3413 \cdot P1 - 1359.8 \cdot I1 = 0 \)
- \( 1267.8 \cdot \text{HE2} + 1251.4 \cdot \text{LE2} + 3413 \cdot P2 - 1359.8 \cdot I2 = 0 \)

- \( X6 + X4 - X2 = 0 \)
- \( X5 + X6 + X9 - X12 = 0 \)
- \( X1 + X2 + X8 - X10 = 0 \)
- \( X7 + X11 + X12 - X10 = 0 \)
- \( X5 + X3 + X7 - X1 = 0 \)
- \( X3 + X4 + X8 - X9 - X11 = 0 \)
- \( 1267.8 \cdot X3 + 1251.4 \cdot X5 + 192 \cdot X7 + 3413 \cdot X13 - 1359.8 \cdot X1 = 0 \)
- \( 1267.8 \cdot X4 + 1251.4 \cdot X6 + 3413 \cdot X14 - 1359.8 \cdot X2 = 0 \)
Step 5: Write Linear Equality Matrix, Vector

Equation: \( \text{Aeq } X = \text{beq} \)

8 equations in 16 variables: Aeq is 8 by 16, beq is 8 by 1

\[
\text{Aeq} = \text{zeros}(8, 16); \text{beq} = \text{zeros}(8, 1);
\]

- \( X_6 + X_4 - X_2 = 0 \)
  \[
  \text{Aeq}(1, [6, 4, 2]) = [1, 1, -1];
  \]
- \( X_5 + X_6 + X_9 - X_{12} = 0 \)
  \[
  \text{Aeq}(2, [5, 6, 9, 12]) = [1, 1, 1, -1];
  \]
- \( X_1 + X_2 + X_8 - X_{10} = 0 \)
  \[
  \text{Aeq}(3, [1, 2, 8, 10]) = [1, 1, 1, -1];
  \]
- \( X_7 + X_{11} + X_{12} - X_{10} = 0 \)
  \[
  \text{Aeq}(4, [7, 11, 12, 10]) = [1, 1, 1, -1];
  \]
- \( X_5 + X_3 + X_7 - X_1 = 0 \)
  \[
  \text{Aeq}(5, 3:2:7) = 1; \text{Aeq}(5, 1) = -1;
  \]
- \( X_3 + X_4 + X_8 - X_9 - X_{11} = 0 \)
  \[
  \text{Aeq}(6, [3, 4, 8, 9, 11]) = [1, 1, 1, -1, -1];
  \]
- \( 1267.8 \cdot X_3 + 1251.4 \cdot X_5 + 192 \cdot X_7 + 3413 \cdot X_{13} - 1359.8 \cdot X_1 = 0 \)
  \[
  \text{Aeq}(7, [3, 5, 7, 13, 1]) = [1267.8, 1251.4, 192, 3413, -1359.8];
  \]
- \( 1267.8 \cdot X_4 + 1251.4 \cdot X_6 + 3413 \cdot X_{14} - 1359.8 \cdot X_2 = 0 \)
  \[
  \text{Aeq}(8, [4, 6, 14, 2]) = [1267.8, 1251.4, 3413, -1359.8];
  \]
Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector \( (x) \)
3. Write vectors for lower and upper bounds \((lb, ub)\)
4. Write matrix and vector of inequalities \((A, b)\)
5. Write matrix and vector of equalities \((Aeq, beq)\)
6. Write nonlinear constraint function
7. Write the objective \((function, or\ vector\ f)\)
8. Call the solver
Step 7: Write Objective Function Vector

- Cost = 0.002614 \cdot HPS + 0.0239 \cdot PP + 0.009825 \cdot EP
  \[ X_{10} \quad X_{15} \quad X_{16} \]
- Equation: minimize cost = \( f^T X = f(1)X_1 + \ldots + f(16)X_{16} \)

\[
\begin{align*}
f &= \text{zeros}(16,1); \\
f(10) &= 0.002614; \\
f(15) &= 0.0239; \\
f(16) &= 0.009825;
\end{align*}
\]
Step 8: Call Solver and Obtain Solution

\[
[x \ fval] = \text{linprog}(f,A,b,Aeq,beq,lb,ub)
\]

Optimization terminated.

\[
x =
\]

\[
1.0e+005 * \\
1.3633 \\
2.4400 \\
1.2816 \\
1.4338 \\
0.0000 \\
1.0062 \\
0.0817 \\
0.0000 \\
0.0000 \\
3.8033 \\
2.7154 \\
1.0062 \\
0.0625 \\
0.0706 \\
0.1124 \\
0.0076
\]

\[
fval = 1.2703e+003
\]

Cost for operating plant: $1270/hour
Examine the Solution

- **Cost breakdown:**
  - Fuel HPS  $X(10)\cdot f(10) = $994.18/hr
  - Purchased Power PP  $X(15)\cdot f(15) = $268.62/hr
  - Penalty EP  $X(16)\cdot f(16) = $7.47/hr

- **Operating conditions**
  - Turbine 1 I1  $X(1) = 136,329$ lbs/hr (max 192,000)
  - Turbine 2 I2  $X(2) = 244,000$ lbs/hr (max 244,000)
  - Valves BF1 BF2  $X(8) = X(9) = 0$
  - Condensate C  $X7 = 8170$ lbs/hr (max 62,000)
Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers
- Exploring Optimization Solvers in MATLAB
- Key Takeaways
Problem 1: Model Gravity as a Function of Altitude

Problem Statement:
A simple model for the acceleration due to gravity $g$ as a function of altitude $h$ is given by $g(h) = a h + b$, where $a$ and $b$ are unknown coefficients.

Given the following experimental data for $g$ at various values of $h$
$g=\{9.8100, 9.7487, 9.6879, 9.6278, 9.5682\}$ and $h=\{0, 20, 40, 60, 80\}$

Find the coefficients $a$ and $b$ that best fits the model to the experimental data.

Problem Identification:
- This is unconstrained least squares curve fitting problem

Solution:
- Can be solved using "\" in basic MATLAB

```matlab
>> h = [0 20 40 60 80]';
>> Coeffs = h\g;
```
### Find Solver and Syntax

#### Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Objective Type</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Least Squares</th>
<th>Smooth nonlinear</th>
<th>Nonsmooth</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td></td>
<td>n/a (f = const, or min = −∞)</td>
<td>quadprog, Information</td>
<td>lsqcurvefit, lsqnonlin, Information</td>
<td>Fminsearch, Fminunc, Information</td>
<td>fminsearch, *</td>
</tr>
<tr>
<td>Bound</td>
<td></td>
<td>linprog, Information</td>
<td>quadprog, Information</td>
<td>lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information</td>
<td>Fminbnd, fmincon, Fseminf, Information</td>
<td>fminbnd, *</td>
</tr>
<tr>
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<td></td>
<td>linprog, Information</td>
<td>quadprog, Information</td>
<td>lsqlin, Information</td>
<td>Fmincon, fseminf, Information</td>
<td>*</td>
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<tr>
<td>General smooth</td>
<td></td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>Fmincon, fseminf, Information</td>
<td>*</td>
</tr>
<tr>
<td>Discrete</td>
<td></td>
<td>intlinprog, Information</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>
Problem 2: Find Minima of $Y = F(X)$, $X$ has bounds

**Problem Statement:**
Find the minima of the function.
$F(x) = x \cdot \sin(x) + x \cdot \cos(2 \cdot x)$

Given $0 < x < 10$

**Problem Identification:**
- This is a bounded/ constrained one dimensional optimization problem

**Solution:**
- Can be solved using “fminbnd” in basic MATLAB
# Find Solver and Syntax

## Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Objective Type</th>
<th>Smooth nonlinear</th>
<th>Nonsmooth</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>Linear ((f = \text{const, or min } = -\infty))</td>
<td>\text{quadprog, Information}</td>
<td>\text{fminsearch, fminunc, Information}</td>
</tr>
<tr>
<td>Bound</td>
<td>Linear</td>
<td>\text{quadprog, Information}</td>
<td>\text{quadprog, Information}</td>
</tr>
<tr>
<td>Linear</td>
<td>Linear</td>
<td>\text{quadprog, Information}</td>
<td>\text{lsqcurvefit, lsqnonlin, Information}</td>
</tr>
<tr>
<td>General smooth</td>
<td>General smooth</td>
<td>\text{fmincon, Information}</td>
<td>\text{fmincon, Information}</td>
</tr>
<tr>
<td>Discrete</td>
<td>Discrete</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>
Problem 3: Compute Volumetric Efficiency of an Engine

Problem Statement:
Find the optimal values for the manifold pressure ratio and the engine revolutions in RPM, to attain the maximum volumetric efficiency.

<table>
<thead>
<tr>
<th>Engine Speed (RPM)</th>
<th>Manifold Pressure Ratio</th>
<th>Volumetric Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3500</td>
<td>0.5</td>
<td>0.896</td>
</tr>
<tr>
<td>3500</td>
<td>0.6</td>
<td>0.915</td>
</tr>
<tr>
<td>3500</td>
<td>0.7</td>
<td>0.929</td>
</tr>
<tr>
<td>3500</td>
<td>0.8</td>
<td>0.887</td>
</tr>
<tr>
<td>3500</td>
<td>0.75</td>
<td>0.937</td>
</tr>
<tr>
<td>3000</td>
<td>0.75</td>
<td>0.862</td>
</tr>
<tr>
<td>4000</td>
<td>0.75</td>
<td>0.913</td>
</tr>
</tbody>
</table>

Problem Identification:
- This is a non-smooth unconstrained multivariable optimization problem

Solution:
- Can be solved using “fminsearch” in Optimization Toolbox

Optimal Params
<table>
<thead>
<tr>
<th>Engine Speed (RPM)</th>
<th>Manifold Pressure Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4300</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Best Value
<table>
<thead>
<tr>
<th>Volumetric Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.973</td>
</tr>
</tbody>
</table>
### Find Solver and Syntax

#### Solvers by Objective and Constraint

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Least Squares</th>
<th>Smooth nonlinear</th>
<th>Nonsmooth</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>n/a (<strong>f</strong> = const, or min = (-\infty))</td>
<td>quadprog, Information</td>
<td>(l_{sqcurvefit}, l_{sqnonlin}, Information)</td>
<td>fminsearch, fminunc, Information</td>
<td>fminsearch, *</td>
</tr>
<tr>
<td>Bound</td>
<td>linprog, Information</td>
<td>quadprog, Information</td>
<td>(l_{sqcurvefit}, l_{sqlin}, l_{sqnonlin}, l_{sqnonneg}, Information)</td>
<td>fminbnd, fmincon, fseminf, Information</td>
<td>fminbnd, *</td>
</tr>
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<td>quadprog, Information</td>
<td>(l_{sqlin}, Information)</td>
<td>fmincon, fseminf, Information</td>
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</tr>
<tr>
<td>General smooth</td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>fmincon, Information</td>
<td>fmincon, fseminf, Information</td>
<td>*</td>
</tr>
<tr>
<td>Discrete</td>
<td>intlinprog, Information</td>
<td>*</td>
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</tbody>
</table>
Problem 4: Hydroelectric Dam Optimization
Equations

Control Volume for Reservoir:

1. \( \text{Storage}(t) = \text{Storage}(t - 1) + ... \)

   \( ... \Delta t \ast [\text{inFlow}(t - 1) - \text{spillFlow}(t - 1) - \text{turbineFlow}(t - 1)] \)

Equation for Electricity:

2. \( \text{Electricity}(t) = \text{TurbineFlow}(t - 1) \ast \left[ \frac{1}{2} k_1 (\text{Storage}(t) + \text{Storage}(t - 1)) + k_2 \right] \)
Decision Variables

- Turbine Flow at each time step
- Spill Flow at each time step

\[
x = \begin{pmatrix}
TF_0 \\
TF_1 \\
\vdots \\
TF_n \\
SF_0 \\
SF_1 \\
\vdots \\
SF_n
\end{pmatrix}
\]
Constraints

1. $0 \leq turbineFlow \leq 25000$ CFS
2. $0 \leq spillFlow$
3. $projectFlow \geq 500$ CFS
4. $projectFlow(t) - projectFlow(t-1) \leq 500$ CFS
5. $50000 \leq storage \leq 100000$ AF
6. $storage_{t=\text{end}} = storage_{t=0}$
Objective Formulation

(1) Quadratic function:

\[ f = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{4} x_1 x_2 + \frac{1}{4} x_1 - \frac{1}{2} x_2 \]
Objective Formulation

(1) Quadratic function:

\[ f = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{4}x_1x_2 + \frac{1}{4}x_1 - \frac{1}{2}x_2 \]

(2) Gradient:

\[ \frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4}x_2 + \frac{1}{4} \]

\[ \frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4}x_1 - \frac{1}{2} \]
Objective Formulation

(1) **Quadratic function:**

\[ f = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{4} x_1 x_2 + \frac{1}{4} x_1 - \frac{1}{2} x_2 \]

(2) **Gradient:**

\[
\frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4} x_2 + \frac{1}{4} \\
\frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4} x_1 - \frac{1}{2}
\]

(3) **Plug in zeros for x:**

\[
c_{x_1} = \frac{1}{4} \\
c_{x_2} = -\frac{1}{2}
\]
### Find Solver and Syntax

#### Solvers by Objective and Constraint

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<td>quadprog, Information</td>
<td>( \backslash ), lsqlin, lsqnonlin, Information</td>
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</tbody>
</table>
## Demo: Scaling Up to a Larger Problem

### Time to Run Optimization (seconds)

<table>
<thead>
<tr>
<th>Decision Variables</th>
<th>FMINCON (no gradient)</th>
<th>FMINCON (with Hessian)</th>
<th>QUADPROG interior-point</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>16</td>
<td>0.46</td>
<td>0.04</td>
</tr>
<tr>
<td>200</td>
<td>238</td>
<td>1.24</td>
<td>0.16</td>
</tr>
<tr>
<td>400</td>
<td>965</td>
<td>3.17</td>
<td>0.77</td>
</tr>
<tr>
<td>800</td>
<td>6083</td>
<td>43</td>
<td>5</td>
</tr>
<tr>
<td>1600</td>
<td>-</td>
<td>*</td>
<td>30</td>
</tr>
<tr>
<td>3200</td>
<td>-</td>
<td>*</td>
<td>190</td>
</tr>
<tr>
<td>6400</td>
<td>-</td>
<td>*</td>
<td>1641</td>
</tr>
</tbody>
</table>

*Gradient/Hessian generation took several hours
Optimization toolboxes support different problem types

<table>
<thead>
<tr>
<th>Objective Types</th>
<th>Constraint Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>Unconstrained</td>
</tr>
<tr>
<td>Quadratic</td>
<td>Bound</td>
</tr>
<tr>
<td>Sum-of-squares</td>
<td>Linear</td>
</tr>
<tr>
<td>(Least Squares)</td>
<td>General Smooth</td>
</tr>
<tr>
<td>Smooth nonlinear</td>
<td>Discrete (integer)</td>
</tr>
<tr>
<td>Nonsmooth</td>
<td></td>
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</tbody>
</table>
Optimization problems

Linear
• LINPROG

Mixed Integer
• INTLINPROG

Quadratic
• QUADPROG

Nonlinear
• FMINCON
• MULTISTART
• GLOBALSEARCH
• PATTERNSEARCH

Least Squares
• LSQLIN
• LSQNONNEG
• LSQCURVEFIT
• LSQNONLIN

Nonsmooth or Noisy
• PATTERNSEARCH
• GA
• SIMULANNEALBND

Multiobjective
• GAMULTIOBJ
Key takeaways

1. Solve Wide Variety of Problems
   - Linear, quadratic, nonlinear, least squares (Optimization Toolbox)
   - Nonlinear, nonsmooth, stochastic, noisy (Global Optimization Toolbox)

2. Set up, Run, and Monitor Optimizations
   - Optimization App
   - MATLAB functions
   - Plot functions, command line display

3. MATLAB Environment
   - Integrated Numerics, Graphics, Symbolic Math
   - Parallel Computing