Numerical Optimization Using MATLAB

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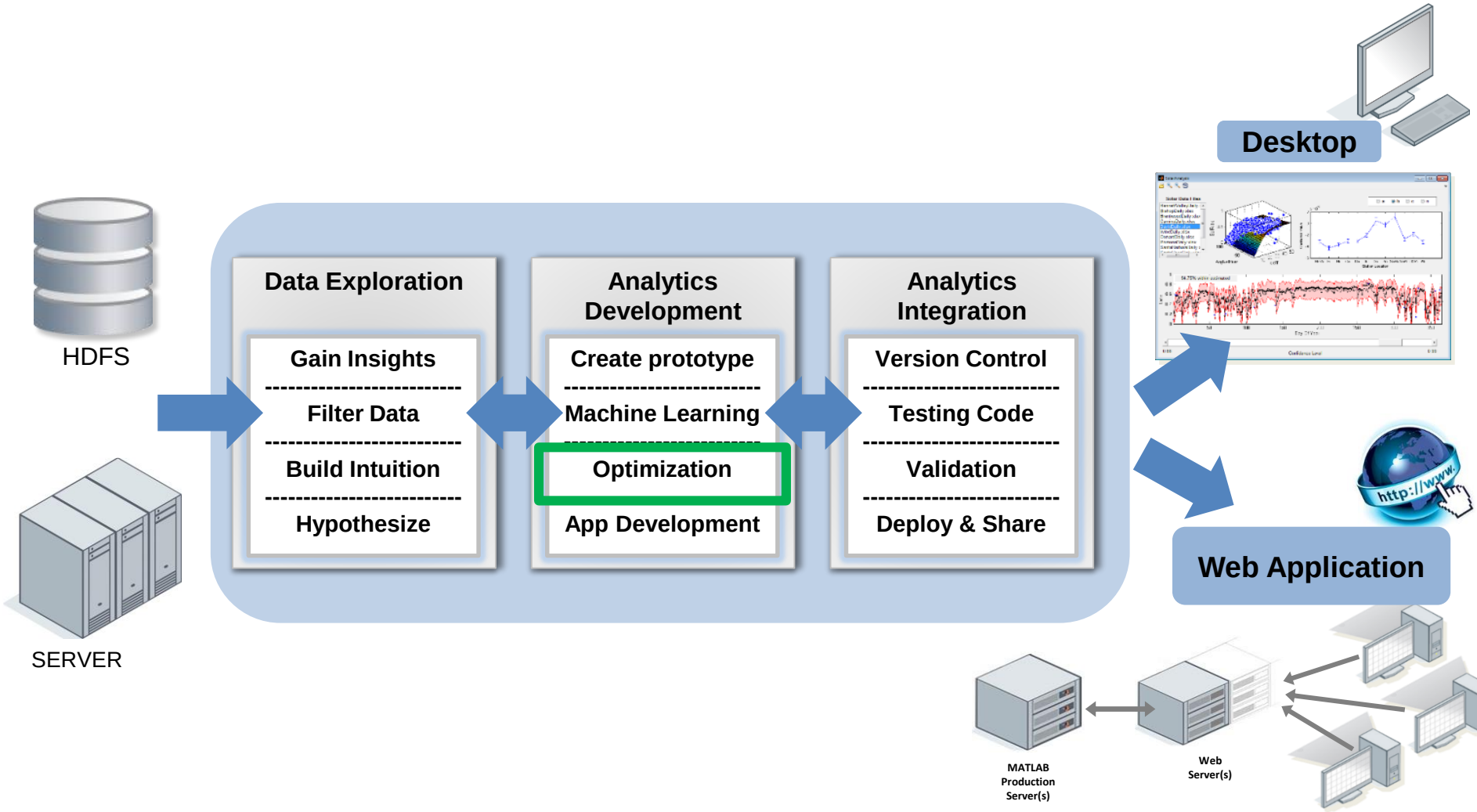
Data Analytics and Technical Computing Workflow



Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers
- Exploring Optimization Solvers in MATLAB
- Key Takeaways

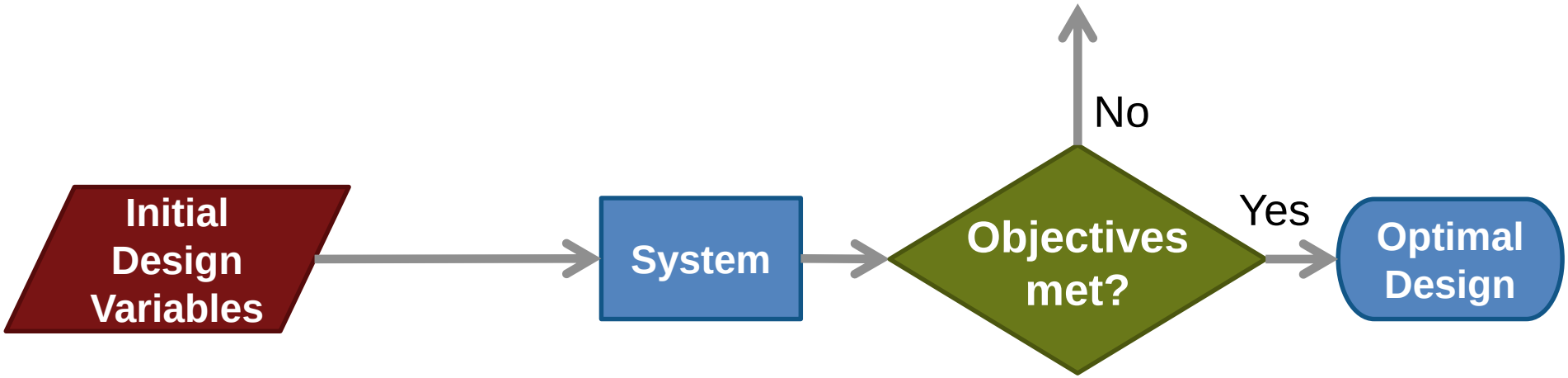
Numerical Optimization Workflow

System

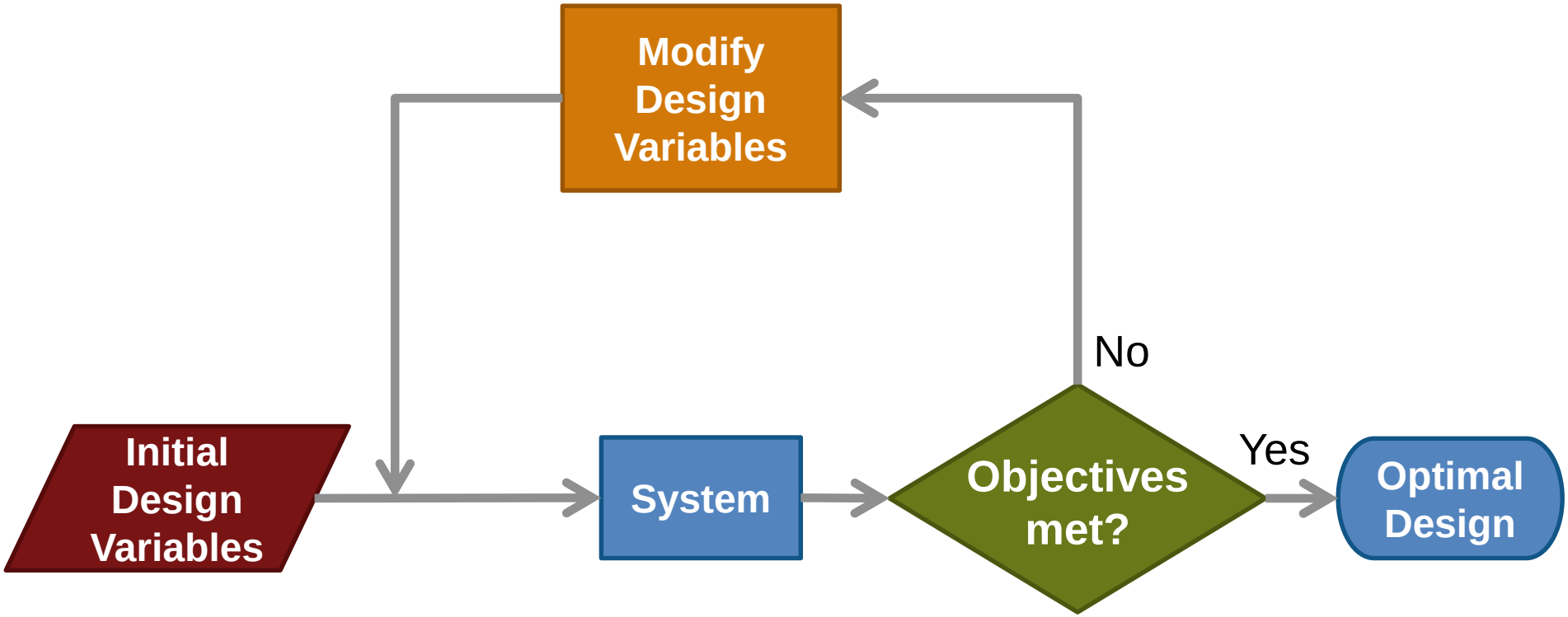
Numerical Optimization Workflow

Numerical Optimization Workflow

Numerical Optimization Workflow



Numerical Optimization Workflow



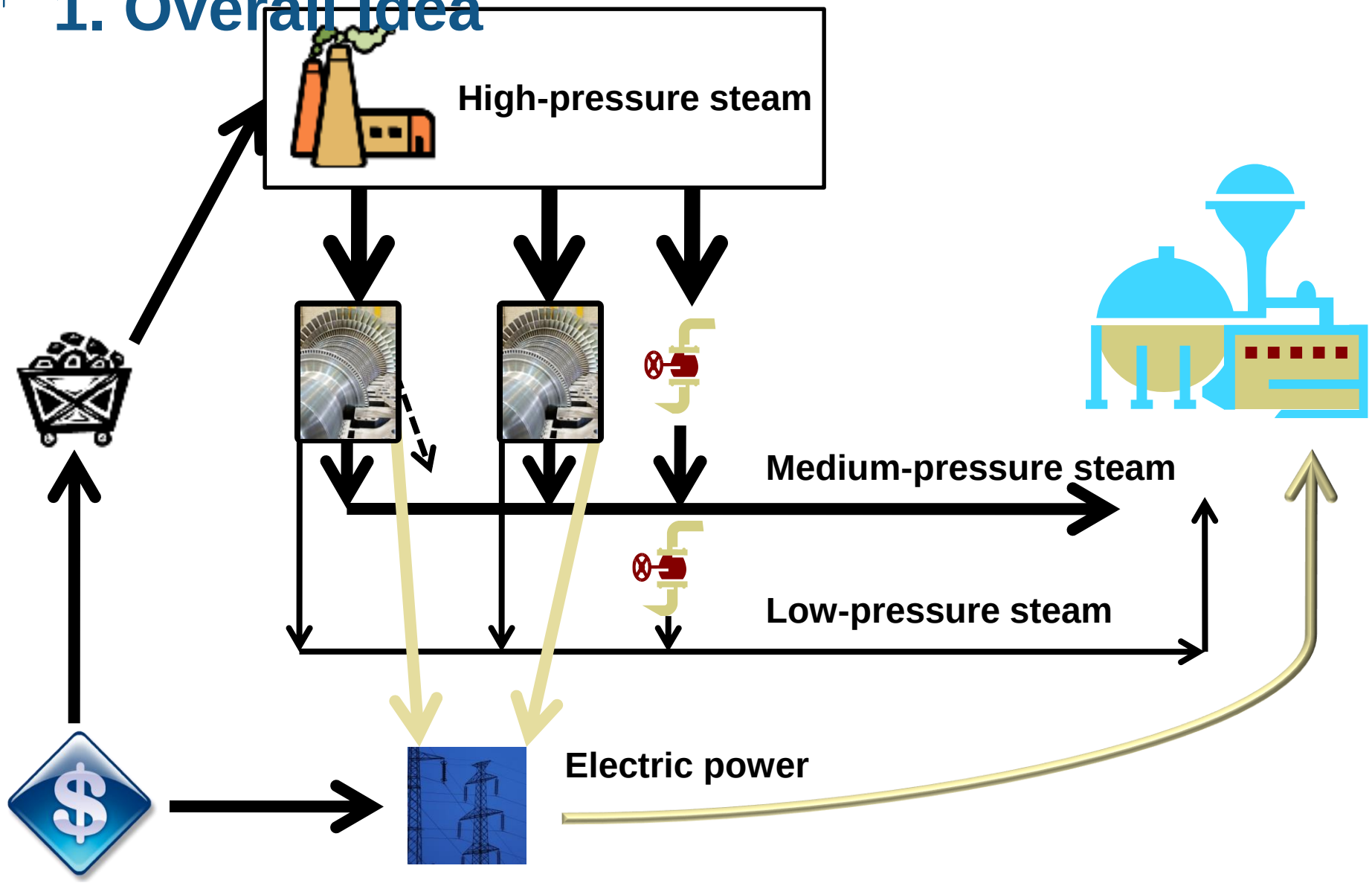
What Is Modeling?

- Modeling: process of turning a problem into a mathematical statement
- Problem can be words, pictures, etc.
- Modeling includes deciding what to keep, and naming variables
- Modeling is an art, not a science

Steps in Modeling

1. Get an overall idea of the problem
2. What is the goal? What are you trying to achieve?
3. Identify variables
4. Identify constraints
5. Identify the inputs and outputs you can control
6. Specify all quantities mathematically
7. Check the model for completeness and correctness

1. Overall Idea



Model adapted from “Optimization of Chemical Processes” by Edgar and Himmelblau, McGraw-Hill, 1988

1. Overall Idea

Costs

- Fuel: $0.002614/lb high-pressure steam
- Electricity purchased: $0.0239/kWh
- Penalty for purchasing less than 12,000 kW: $0.009825/kWh

Demands

- Medium-pressure steam: 271,536 lb/h
- Low-pressure steam: 100,623 lb/h
- Electric power: 24,550 kW

2. What Is the Goal?

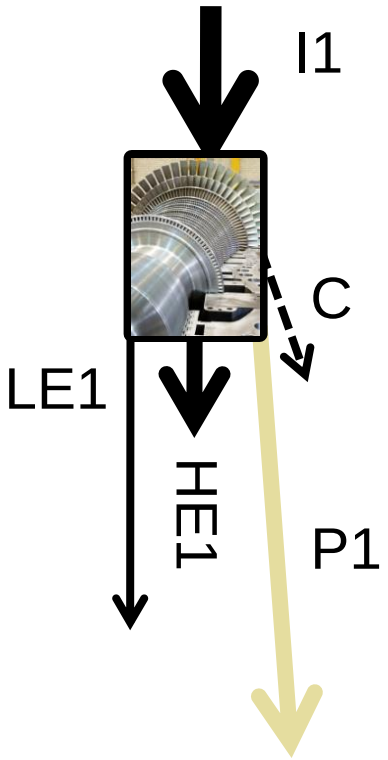
Objective

- Run the plant at minimal cost
- In other words, satisfy the constraints using the least amount of money

3. Identify Variables

Turbine 1 Model

- Identify Variables
  - I1 = inlet flow rate
  - C = condensate flow rate
  - LE1 = low-pressure steam flow rate
  - HE1 = medium-pressure steam flow rate
  - P1 = power generated by Turbine 1



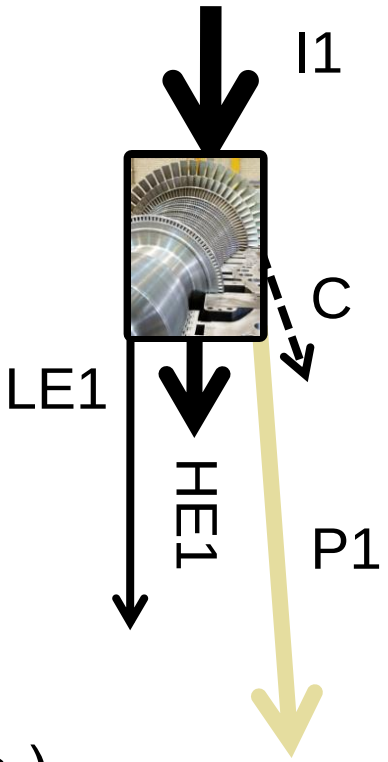
Units

Flow rate: lb/h

Power: kW

4. Identify Constraints
Turbine 1 Model

- $2500 \leq P1 \leq 6250$ (capacity)
- $I1 \leq 192{,}000$ (max inlet flow)
- $C \leq 62{,}000$ (max condensate)
- $I1 - HE1 \leq 132{,}000$ (max internal)
- $I1 = LE1 + HE1 + C$ (mass conservation)
- $1359.8 \cdot I1 = 1267.8 \cdot HE1 + 1251.4 \cdot LE1 + 192 \cdot C + 3413 \cdot P1$ (energy cons.)

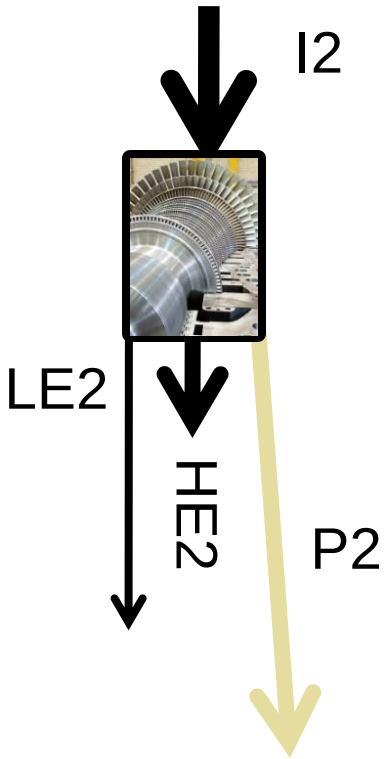


Flow rate lb/h, **Power** kW, **Energy** BTU

3&4. Identify Variables/Constraints

Turbine 2 Model

- $3000 \leq P2 \leq 9000$
- $I2 \leq 244{,}000$ (max inlet flow)
- $LE2 \leq 142{,}000$ (max low-pressure flow)
- $I2 = LE2 + HE2$ (mass conservation)
- $1359.8 \cdot I2 = 1267.8 \cdot HE2 + 1251.4 \cdot LE2 + 3413 \cdot P2$ (energy conservation)



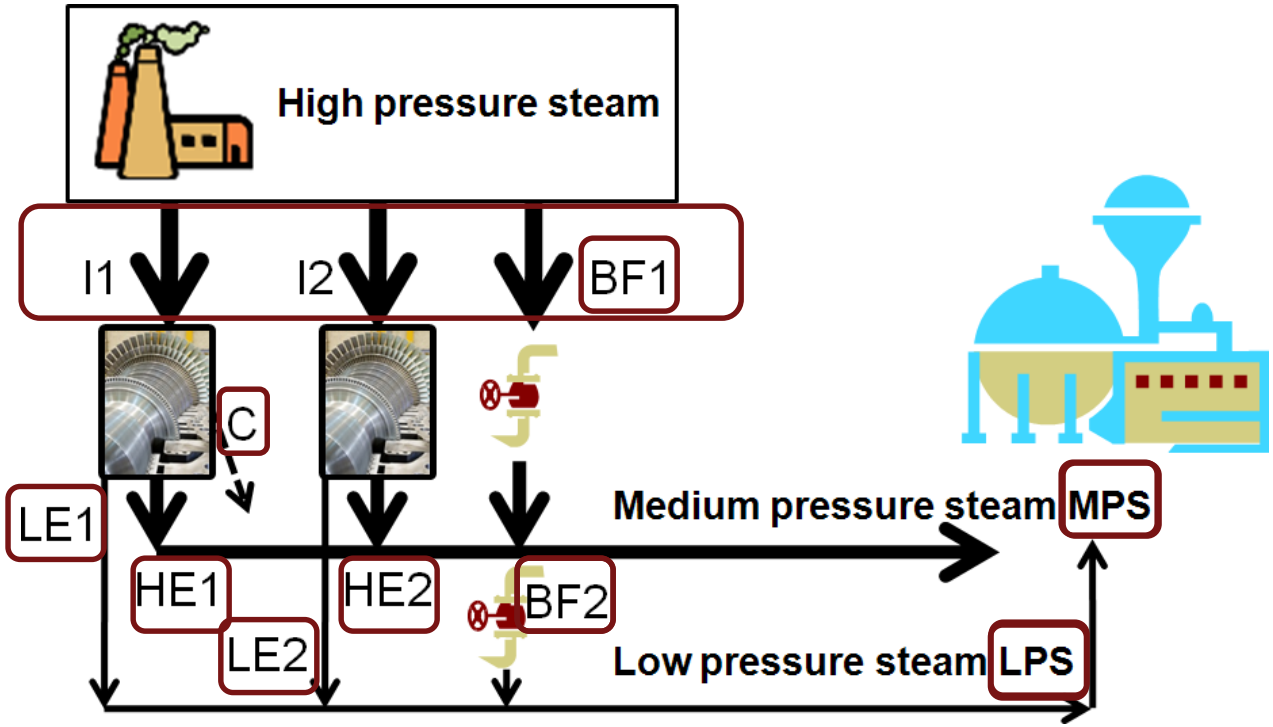
Flow rate lb/h, **Power** kW, **Energy** BTU

4. Identify Constraints

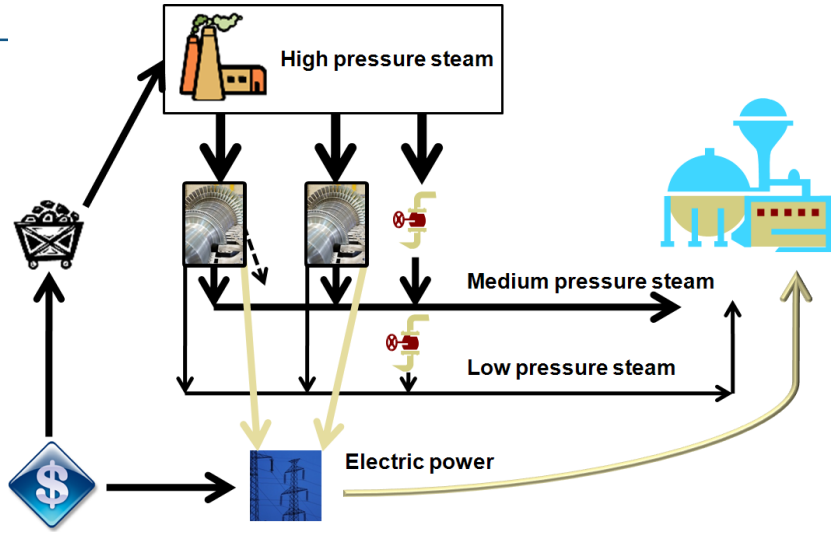
Material Balance and Demands

- HPS = high-pressure steam (MPS, LPS)
- BFi = flow through valve i

- HPS = I1 + I2 + BF1
- HPS = C + MPS + LPS
- MPS = HE1 + HE2 + BF1 – BF2
- LPS = LE1 + LE2 + BF2
- MPS ≥ 271,536
- LPS ≥ 100,623



4. Identify Constraints
Electric Power Expressions



- PP = purchased power
- EP = excess power (unpurchased)

- $P1 + P2 + PP \geq 24{,}550$
- $EP + PP \geq 12{,}000$

5 – 7 Model Summary – Collected Equations

Turbine 1

- $2500 \le P1 \le 6250$
- $I1 \le 192{,}000$
- $C \le 62{,}000$
- $I1 - HE1 \le 132{,}000$
- $I1 = LE1 + HE1 + C$
- $1359.8 \cdot I1 = 1267.8 \cdot HE1 + 1251.4 \cdot LE1 + 192 \cdot C + 3413 \cdot P1$

Turbine 2

- $3000 \le P2 \le 9000$
- $I2 \le 244{,}000$
- $LE2 \le 142{,}000$
- $I2 = LE2 + HE2$
- $1359.8 \cdot I2 = 1267.8 \cdot HE2 + 1251.4 \cdot LE2 + 3413 \cdot P2$

Conserved Mass

- $HPS = I1 + I2 + BF1$
- $HPS = C + MPS + LPS$
- $LPS = LE1 + LE2 + BF2$
- $MPS = HE1 + HE2 + BF1 - BF2$

Requirements

- $P1 + P2 + PP \ge 24{,}550$
- $EP + PP \ge 12{,}000$
- $MPS \ge 271{,}536$
- $LPS \ge 100{,}623$
- All variables positive

- **$Cost = 0.002614 \cdot HPS + 0.0239 \cdot PP + 0.009825 \cdot EP$**

Find Solver and Syntax

Solvers by Objective and Constraint

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Nonsmooth
None	n/a (f = const, or min = $-\infty$)	quadprog, Information	\, lsqcurvefit, lsqnonlin, Information	fminsearch, fminunc, Information	fminsearch, *
Bound	linprog, Information	quadprog, Information	lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information	fminbnd, fmincon, fseminf, Information	fminbnd, *
Linear	linprog, Information	quadprog, Information	lsqlin, Information	fmincon, fseminf, Information	*
General smooth	fmincon, Information	fmincon, Information	fmincon, Information	fmincon, fseminf, Information	*
Discrete	intlinprog, Information	*	*	*	*

Find Solver and Syntax

- linprog:

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \le b, \\ Aeq \cdot x = beq, \\ lb \le x \le ub. \end{cases}$$

- Syntax:

```
[x fval] = linprog(f,A,b,Aeq,beq,lb,ub)
```

- Syntax implies linear inequalities, linear equalities, and bounds should be separated

Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector (x)
3. Write vectors for lower and upper bounds (lb, ub)
4. Write matrix and vector of inequalities (A, b)
5. Write matrix and vector of equalities (Aeq, beq)
6. Write nonlinear constraint function
7. Write the objective (function, or vector f)
8. Call the solver

Step 1: Separate Bounds — Inequalities Involving One Variable

- $2500 \leq P1 \leq 6250$
- $I1 \leq 192{,}000$
- $C \leq 62{,}000$
- $3000 \leq P2 \leq 9000$
- $I2 \leq 244{,}000$
- $LE2 \leq 142{,}000$
- $MPS \geq 271{,}536$
- $LPS \geq 100{,}623$
- All variables are positive

Step 1: Separate Linear Constraints (In)Equalities Involving Two+ Variables

- $I1 - HE1 \leq 132{,}000$
- $I2 = LE2 + HE2$
- $LPS = LE1 + LE2 + BF2$
- $EP + PP \geq 12{,}000$
- $HPS = I1 + I2 + BF1$
- $HPS = C + MPS + LPS$
- $I1 = LE1 + HE1 + C$
- $MPS = HE1 + HE2 + BF1 - BF2$
- $1359.8 \cdot I1 = 1267.8 \cdot HE1 + 1251.4 \cdot LE1 + 192 \cdot C + 3413 \cdot P1$
- $1359.8 \cdot I2 = 1267.8 \cdot HE2 + 1251.4 \cdot LE2 + 3413 \cdot P2$
- $P1 + P2 + PP \geq 24{,}550$

Step 2: Combine Variables into One Vector

$$\mathbf{X_k = X(k)}$$

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 3:
Write the Lower Bounds Vector

- All variables positive
- $2500 \leq$ P1 (X_{13})
- $3000 \leq$ P2 (X_{14})
- $271{,}536 \leq$ MPS (X_{11})
- $100{,}623 \leq$ LPS (X_{12})

```
lb(13) = 2500;
lb(14) = 3000;
lb(11) = 271536;
lb(12) = 100623;
```

```
lb = zeros(16,1);
```

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 3:
Write the Upper Bounds Vector

```
ub = Inf(16,1);
```

- P1 ≤ 6250 (X_{13}) `ub(13) = 6250;`
- I1 ≤ 192,000 (X_1) `ub(1) = 192000;`
- C ≤ 62,000 (X_7) `ub(7) = 62000;`
- P2 ≤ 9000 (X_{14}) `ub(14) = 9000;`
- I2 ≤ 244,000 (X_2) `ub(2) = 244000;`
- LE2 ≤ 142,000 (X_6) `ub(6) = 142000;`

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 4: Write Linear Inequality Matrix and Vector

$$A \cdot x \leq b$$

- I1 – HE1 ≤ 132,000
- EP + PP ≥ 12,000
- P1 + P2 + PP ≥ 24,550
- Put into "less than" form:
- I1 – HE1 ≤ 132,000;
- –EP – PP ≤ – 12,000;
- –P1 – P2 – PP ≤ – 24,550;

$$X_1 - X_3 \leq 132{,}000$$

$$-X_{16} - X_{15} \leq -12{,}000$$

$$-X_{13} - X_{14} - X_{15} \leq -24{,}550$$

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 4: Write Linear Inequality Matrix and Vector

$$A \cdot x \leq b$$

- $X_1 - X_3 \leq 132{,}000$
- $-X_{16} - X_{15} \leq -12{,}000$
- $-X_{13} - X_{14} - X_{15} \leq -24{,}550$
- Matrix A and vector b for $AX \leq b$
- There are three inequalities in 16 variables
- Therefore A should have 3 rows and 16 columns
- b should be a column vector with 3 entries

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 4: Write Linear Inequality Matrix and Vector

$$A \cdot x \leq b$$

- $X_1 - X_3 \leq 132{,}000$
- $-X_{16} - X_{15} \leq -12{,}000$
- $-X_{13} - X_{14} - X_{15} \leq -24{,}550$
- Matrix A and vector b for $AX \leq b$

```
A = zeros(3,16);
A(1,1) = 1; A(1,3) = -1; b(1) = 132000;
A(2,15:16) = -1; b(2) = -12000;
A(3,13:15) = -1; b(3) = -24550;
```

Step 5: Write Linear Equality Matrix and Vector

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

- I2 = LE2 + HE2
- LPS = LE1 + LE2 + BF2
- HPS = I1 + I2 + BF1
- HPS = C + MPS + LPS
- I1 = LE1 + HE1 + C
- MPS = HE1 + HE2 + BF1 – BF2
- 1359.8·I1 = 1267.8·HE1 + 1251.4·LE1 + 192·C + 3413·P1
- 1359.8·I2 = 1267.8·HE2 + 1251.4·LE2 + 3413·P2

Step 5: Write Linear Equality Matrix and Vector

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

- LE2 + HE2 – I2 = 0
- LE1 + LE2 + BF2 – LPS = 0
- I1 + I2 + BF1 – HPS = 0
- C + MPS + LPS – HPS = 0
- LE1 + HE1 + C – I1 = 0
- HE1 + HE2 + BF1 – BF2 – MPS = 0
- 1267.8·HE1 + 1251.4·LE1 + 192·C + 3413·P1 – 1359.8·I1 = 0
- 1267.8·HE2 + 1251.4·LE2 + 3413·P2 – 1359.8·I2 = 0

- $X_6 + X_4 - X_2 = 0$
- $X_5 + X_6 + X_9 - X_{12} = 0$
- $X_1 + X_2 + X_8 - X_{10} = 0$
- $X_7 + X_{11} + X_{12} - X_{10} = 0$
- $X_5 + X_3 + X_7 - X_1 = 0$
- $X_3 + X_4 + X_8 - X_9 - X_{11} = 0$
- $1267.8 \cdot X_3 + 1251.4 \cdot X_5 + 192 \cdot X_7 + 3413 \cdot X_{13} - 1359.8 \cdot X_1 = 0$
- $1267.8 \cdot X_4 + 1251.4 \cdot X_6 + 3413 \cdot X_{14} - 1359.8 \cdot X_2 = 0$

Step 5: Write Linear Equality Matrix, Vector

Equation: Aeq X = beq

8 equations in 16 variables: Aeq is 8 by 16, beq is 8 by 1

```
Aeq = zeros(8,16); beq = zeros(8,1);
```

- $X_6 + X_4 - X_2 = 0$

```
Aeq(1,[6,4,2]) = [1,1,-1];
```

- $X_5 + X_6 + X_9 - X_{12} = 0$

```
Aeq(2,[5,6,9,12]) = [1,1,1,-1];
```

- $X_1 + X_2 + X_8 - X_{10} = 0$

```
Aeq(3,[1,2,8,10]) = [1,1,1,-1];
```

- $X_7 + X_{11} + X_{12} - X_{10} = 0$

```
Aeq(4,[7,11,12,10]) =
[1,1,1,-1];
```

- $X_5 + X_3 + X_7 - X_1 = 0$

```
Aeq(5,3:2:7)=1; Aeq(5,1)=-1;
```

- $X_3 + X_4 + X_8 - X_9 - X_{11} = 0$

```
Aeq(6,[3,4,8,9,11]) =
[1,1,1,-1,-1];
```

- $1267.8 \cdot X_3 + 1251.4 \cdot X_5 + 192 \cdot X_7 + 3413 \cdot X_{13} - 1359.8 \cdot X_1 = 0$

```
Aeq(7,[3,5,7,13,1]) =
[1267.8,1251.4,192,3413,
-1359.8];
```

- $1267.8 \cdot X_4 + 1251.4 \cdot X_6 + 3413 \cdot X_{14} - 1359.8 \cdot X_2 = 0$

```
Aeq(8,[4,6,14,2]) =
[1267.8,1251.4,3413,-1359.8];
```

Steps to Transform Equations

1. Separate bounds, linear equalities, linear inequalities, nonlinear equalities, and nonlinear inequalities
2. Combine all variables into one vector (x)
3. Write vectors for lower and upper bounds (lb, ub)
4. Write matrix and vector of inequalities (A, b)
5. Write matrix and vector of equalities (Aeq, beq)
6. Write nonlinear constraint function
7. Write the objective (function, or vector f)
8. Call the solver

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

Step 7: Write Objective Function Vector

- Cost = $0.002614 \cdot \underset{X_{10}}{HPS} + 0.0239 \cdot \underset{X_{15}}{PP} + 0.009825 \cdot \underset{X_{16}}{EP}$
- Equation: minimize cost = $f^T X = f(1)X_1 + \ldots + f(16)X_{16}$

```
f = zeros(16,1);
f(10) = 0.002614;
f(15) = 0.0239;
f(16) = 0.009825;
```

Step 8: Call Solver and Obtain Solution

```
[x fval] = linprog(f,A,b,Aeq,beq,lb,ub)
```

```
Optimization terminated.


x =

  1.0e+005 *

    1.3633
    2.4400
    1.2816
    1.4338
    0.0000
    1.0062
    0.0817
    0.0000
    0.0000
    3.8033
    2.7154
    1.0062
    0.0625
    0.0706
    0.1124
    0.0076


fval =

  1.2703e+003
```

Cost for operating plant: $1270/hour

Examine the Solution

- X_1 = I1
- X_2 = I2
- X_3 = HE1
- X_4 = HE2
- X_5 = LE1
- X_6 = LE2
- X_7 = C
- X_8 = BF1
- X_9 = BF2
- X_{10} = HPS
- X_{11} = MPS
- X_{12} = LPS
- X_{13} = P1
- X_{14} = P2
- X_{15} = PP
- X_{16} = EP

- Cost breakdown :
  - Fuel HPS X(10)*f(10) = $994.18/hr
  - Purchased Power PP X(15)*f(15) = $268.62/hr
  - Penalty EP X(16)*f(16) = $7.47/hr
- Operating conditions
  - Turbine 1 I1 X(1) = 136,329 lbs/hr (max 192,000)
  - Turbine 2 I2 X(2) = 244,000 lbs/hr (max 244,000)
  - Valves BF1 BF2 X(8) = X(9) = 0
  - Condensate C X7 = 8170 lbs/hr (max 62,000)

Agenda

- Numerical Optimization Workflow
  - Modeling an Optimization Problem
  - Structured Approach for Solver Selection
  - Transforming and Solving Problem using Optimization Solvers
- Exploring Optimization Solvers in MATLAB
- Key Takeaways

Problem 1: Model Gravity as a Function of Altitude

Problem Statement:

A simple model for the acceleration due to gravity g as a function of altitude h is given by **g(h) = a h + b**, where **a and b** are unknown coefficients.

Given the following experimental data for **g** at various values of **h**

g=[9.8100 9.7487 9.6879 9.6278 9.5682] and

h=[0 20 40 60 80]

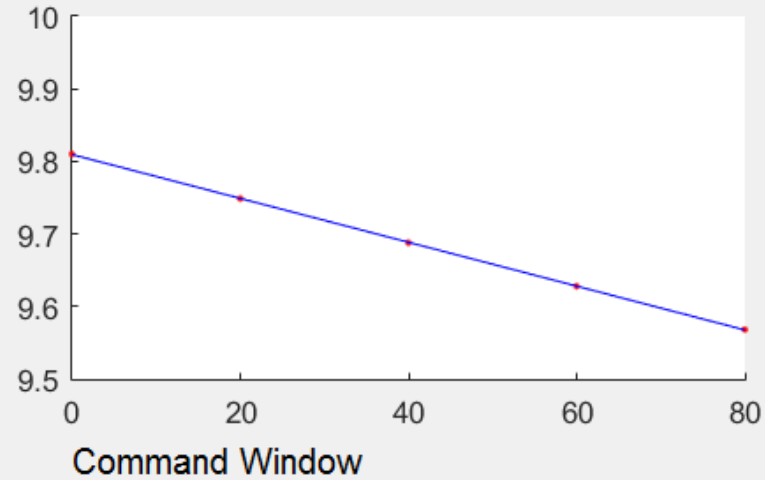
Find the coefficients a and b that best fits the model to the experimental data.

Problem Identification:

- This is unconstrained least squares curve fitting problem

Solution:

- Can be solved using **"\"** in basic MATLAB



Command Window

```
>> g = [9.8100 9.7487 9.6879 9.6278
9.5682]';

>> h = [0 20 40 60 80]';

>> h = [h ones(size(h,1),1)];

>> Coeffs = h\g;
```

Find Solver and Syntax

Solvers by Objective and Constraint

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Nonsmooth
None	n/a (f = const, or min = $-\infty$)	quadprog, Information	\, lsqcurvefit, lsqnonlin, Information	fminsearch, fminunc, Information	fminsearch, *
Bound	linprog, Information	quadprog, Information	lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information	fminbnd, fmincon, fseminf, Information	fminbnd, *
Linear	linprog, Information	quadprog, Information	lsqlin, Information	fmincon, fseminf, nformation	*
General smooth	fmincon, Information	fmincon, Information	fmincon, Information	fmincon, fseminf, nformation	*
Discrete	intlinprog, Information	*	*	*	*

Problem 2: Find Minima of Y = F(X), X has bounds

Problem Statement:

Find the minima of the function.

F(x) = x.*sin(x) + x.*cos(2.*x)

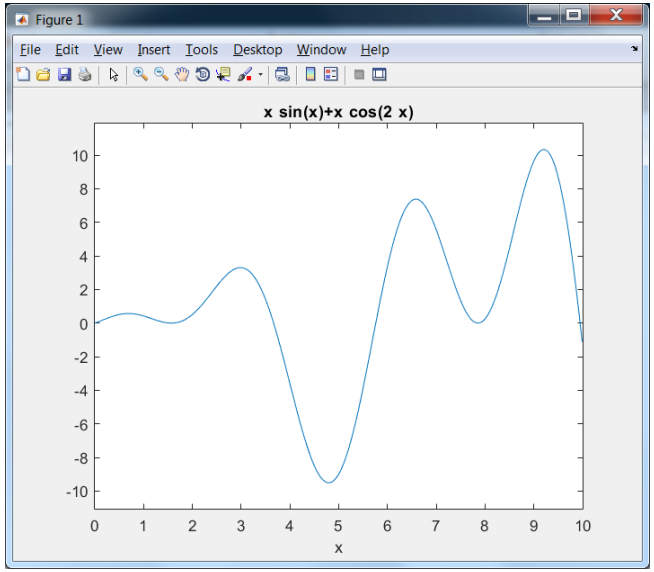
Given 0<x<10

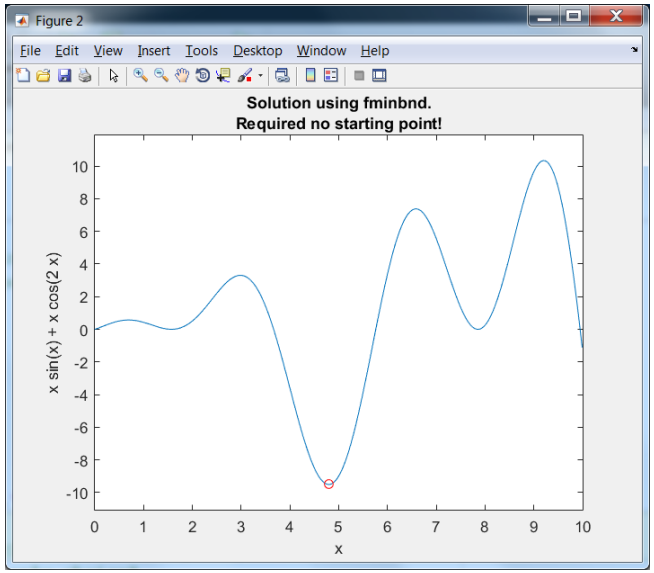
Problem Identification:

- This is a bounded/ constrained one dimensional optimization problem

Solution:

- Can be solved using **“fminbnd”** in basic MATLAB





Find Solver and Syntax

Solvers by Objective and Constraint

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Nonsmooth
None	n/a (f = const, or min = $-\infty$)	quadprog, Information	\, lsqcurvefit, lsqnonlin, Information	fminsearch, fminunc, Information	fminsearch, *
Bound	linprog, Information	quadprog, Information	lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information	fminbnd, fmincon, fseminf, Information	fminbnd, *
Linear	linprog, Information	quadprog, Information	lsqlin, Information	fmincon, fseminf, Information	*
General smooth	fmincon, Information	fmincon, Information	fmincon, Information	fmincon, fseminf, Information	*
Discrete	intlinprog, Information	*	*	*	*

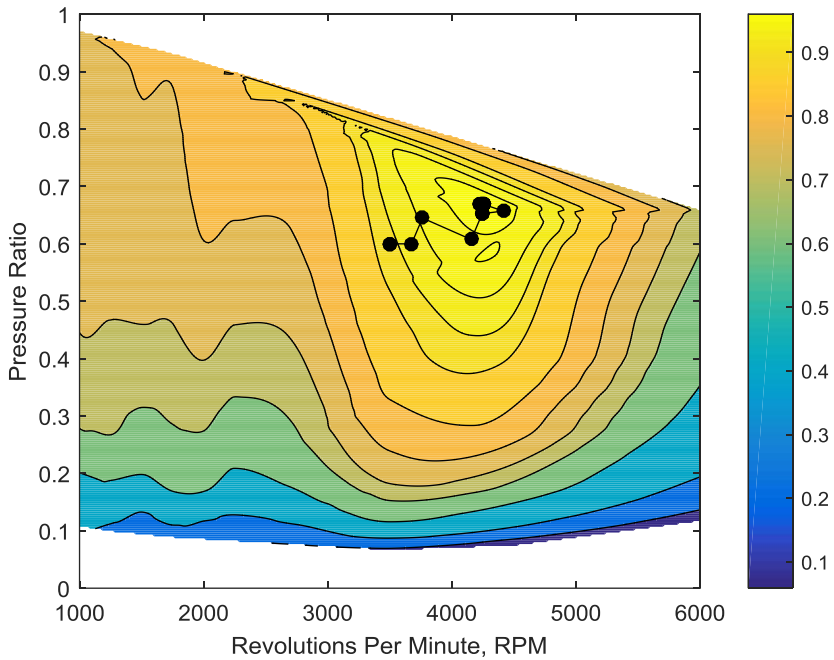
Problem 3: Compute Volumetric Efficiency of an Engine

Problem Statement:

Find the optimal values for the manifold pressure ratio and the engine revolutions in RPM, to attain the maximum volumetric efficiency.

Engine Model

Engine Speed (RPM)	Manifold Pressure Ratio	Volumetric Efficiency
3500	0.5	0.896
3500	0.6	0.915
3500	0.7	0.929
3500	0.8	0.887
3500	0.75	0.937
3000	0.75	0.862
4000	0.75	0.913

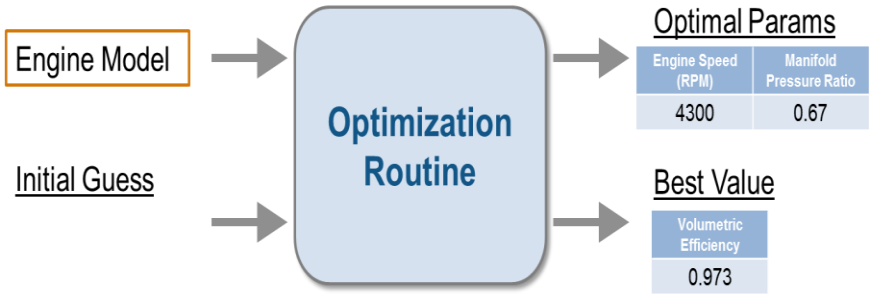


Problem Identification:

- This is a non-smooth unconstrained multivariable optimization problem

Solution:

- Can be solved using **"fminsearch"** in Optimization Toolbox



Find Solver and Syntax

Solvers by Objective and Constraint

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Nonsmooth
None	n/a (f = const, or min = $-\infty$)	quadprog, Information	\, lsqcurvefit, lsqnonlin, Information	fminsearch, fminunc, Information	fminsearch, *
Bound	linprog, Information	quadprog, Information	lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information	fminbnd, fmincon, fseminf, Information	fminbnd, *
Linear	linprog, Information	quadprog, Information	lsqlin, Information	fmincon, fseminf, Information	*
General smooth	fmincon, Information	fmincon, Information	fmincon, Information	fmincon, fseminf, Information	*
Discrete	intlinprog, Information	*	*	*	*

Problem 4: Hydroelectric Dam Optimization

Equations

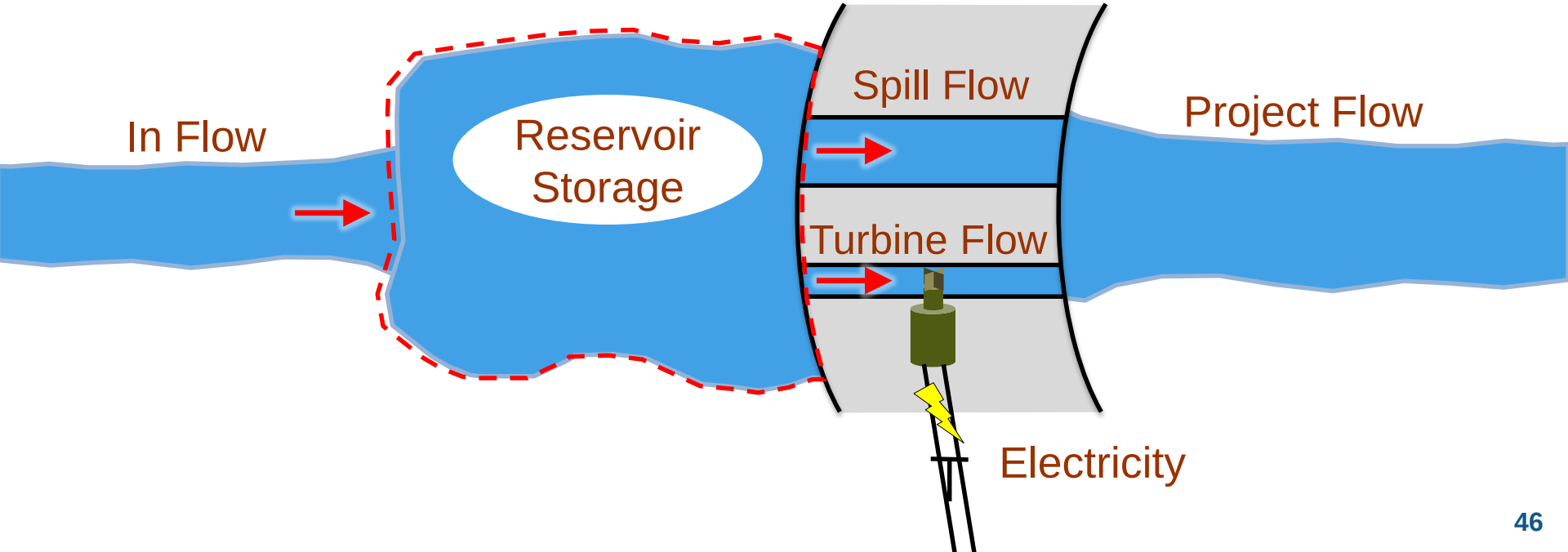
Control Volume for Reservoir:

(1) $\text{Storage}(t) = \text{Storage}(t-1) + \ldots$

$$\ldots\ \Delta t * [inFlow(t-1) - spillFlow(t-1) - turbineFlow(t-1)\ ]$$

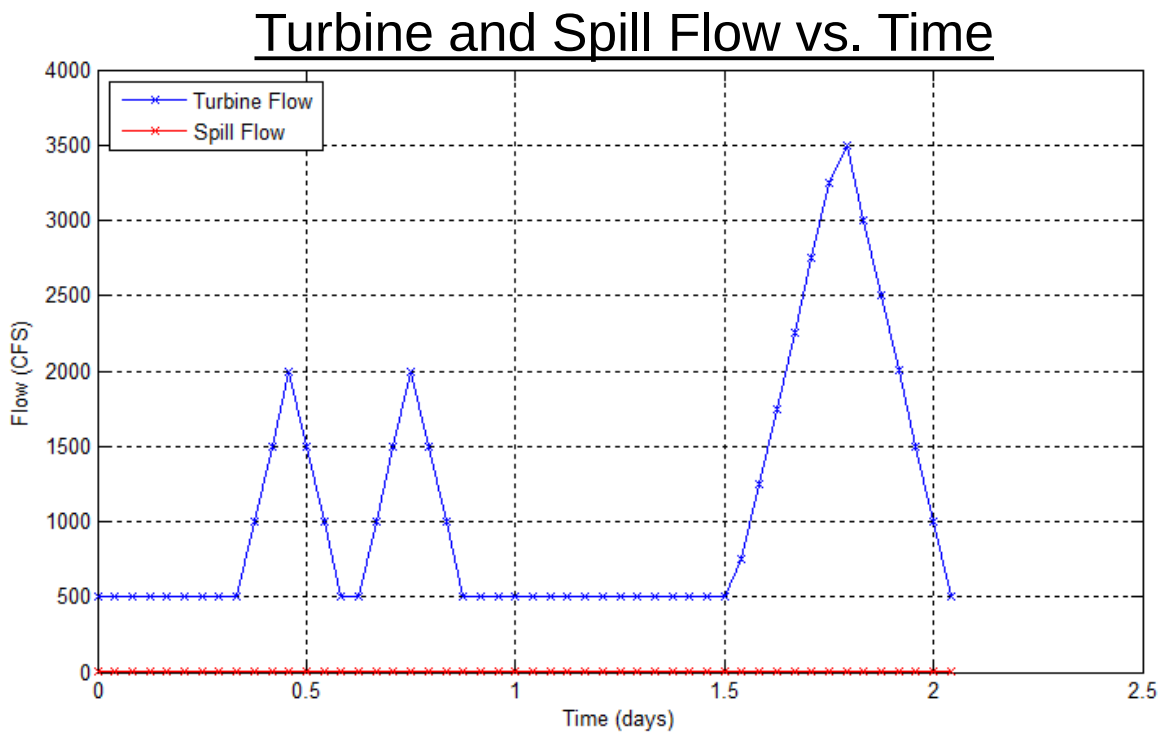
Equation for Electricity:

(2) $\text{Electricity}(t) = \text{TurbineFlow}(t-1) * \left[\frac{1}{2}k_1\big(\text{Storage}(t) + \text{Storage}(t-1)\big) + k_2\right]$



Decision Variables

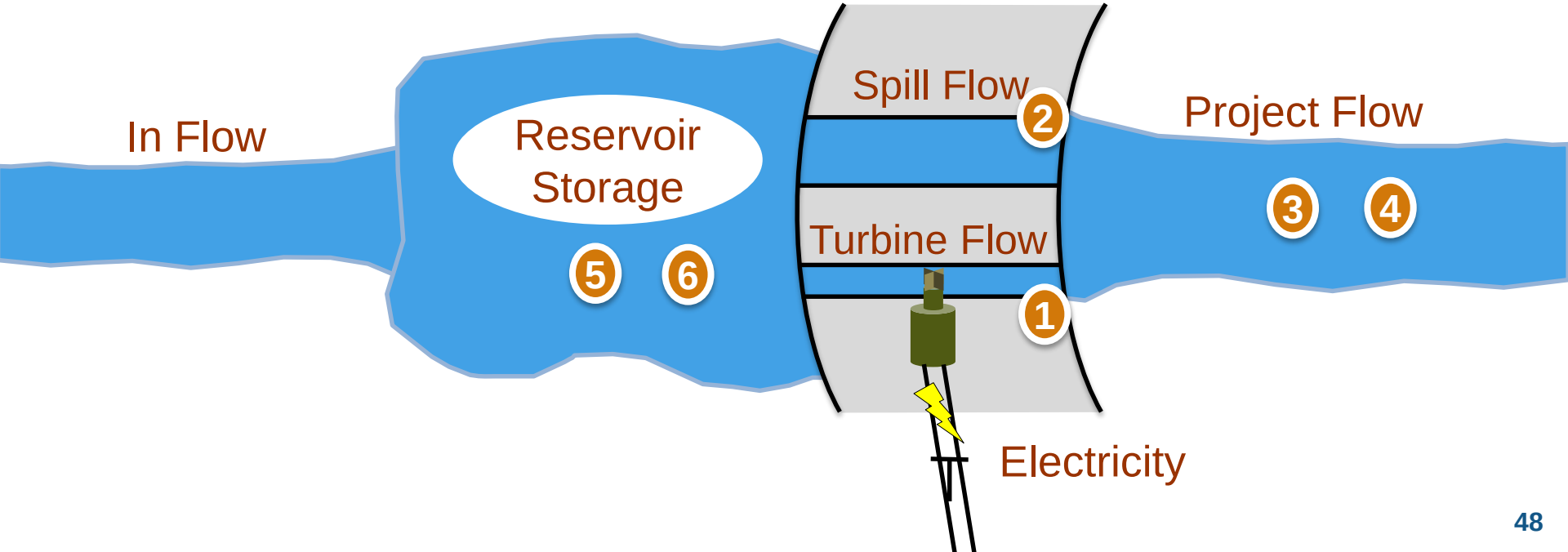
- Turbine Flow at each time step
- Spill Flow at each time step



$$x = \begin{pmatrix} TF_0 \\ TF_1 \\ \vdots \\ TF_n \\ SF_0 \\ SF_1 \\ \vdots \\ SF_n \end{pmatrix}$$

Constraints

1. $0 \leq turbineFlow \leq 25000 \text{ CFS}$
2. $0 \leq spillFlow$
3. $projectFlow \geq 500 \text{ CFS}$
4. $projectFlow(t) - projectFlow(t-1) \leq 500 \text{ CFS}$
5. $50000 \leq storage \leq 100000 \text{ AF}$
6. $storage_{t=end} = storage_{t=0}$



Objective Formulation

(1) <u>Quadratic function:</u>

$$f = \frac{1}{2}x_1{}^2 + \frac{1}{2}x_2{}^2 - \frac{1}{4}x_1x_2 + \frac{1}{4}x_1 - \frac{1}{2}x_2$$

Objective Formulation

(1) <u>Quadratic function:</u>

$$f = \frac{1}{2}x_1{}^2 + \frac{1}{2}x_2{}^2 - \frac{1}{4}x_1x_2 + \frac{1}{4}x_1 - \frac{1}{2}x_2$$

(2) <u>Gradient:</u>

$$\frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4}x_2 + \frac{1}{4}$$

$$\frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4}x_1 - \frac{1}{2}$$

Objective Formulation

(1) Quadratic function:

$$f = \frac{1}{2}x_1{}^2 + \frac{1}{2}x_2{}^2 - \frac{1}{4}x_1x_2 + \frac{1}{4}x_1 - \frac{1}{2}x_2$$

(2) Gradient:

$$\frac{\partial f}{\partial x_1} = x_1 - \frac{1}{4}x_2 + \frac{1}{4}$$

$$\frac{\partial f}{\partial x_2} = x_2 - \frac{1}{4}x_1 - \frac{1}{2}$$

(3) Plug in zeros for x:

$$c_{x_1} = \frac{1}{4}$$

$$c_{x_2} = -\frac{1}{2}$$

Find Solver and Syntax

Solvers by Objective and Constraint

Constraint Type	Objective Type				
	Linear	Quadratic	Least Squares	Smooth nonlinear	Nonsmooth
None	n/a (f = const, or min = $-\infty$)	quadprog, Information	\, lsqcurvefit, lsqnonlin, Information	fminsearch, fminunc, Information	fminsearch, *
Bound	linprog, Information	quadprog, Information	lsqcurvefit, lsqlin, lsqnonlin, lsqnonneg, Information	fminbnd, fmincon, fseminf, Information	fminbnd, *
Linear	linprog, Information	quadprog, Information	lsqlin, Information	fmincon, fseminf, Information	*
General smooth	fmincon, Information	fmincon, Information	fmincon, Information	fmincon, fseminf, Information	*
Discrete	intlinprog, Information	*	*	*	*

Demo: Scaling Up to a Larger Problem

Time to Run Optimization (seconds)

Decision Variables	FMINCON (no gradient)	FMINCON (with Hessian)	QUADPROG interior-point
100	16	0.46	0.04
200	238	1.24	0.16
400	965	3.17	0.77
800	6083	43	5
1600	-	*	30
3200	-	*	190
6400	-	*	1641

*Gradient/Hessian generation took several hours

Optimization toolboxes support different problem types

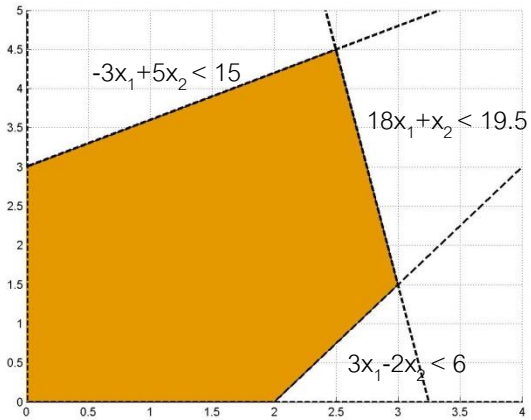
Objective Types

- Linear
- Quadratic
- Sum-of-squares (Least Squares)
- Smooth nonlinear
- Nonsmooth

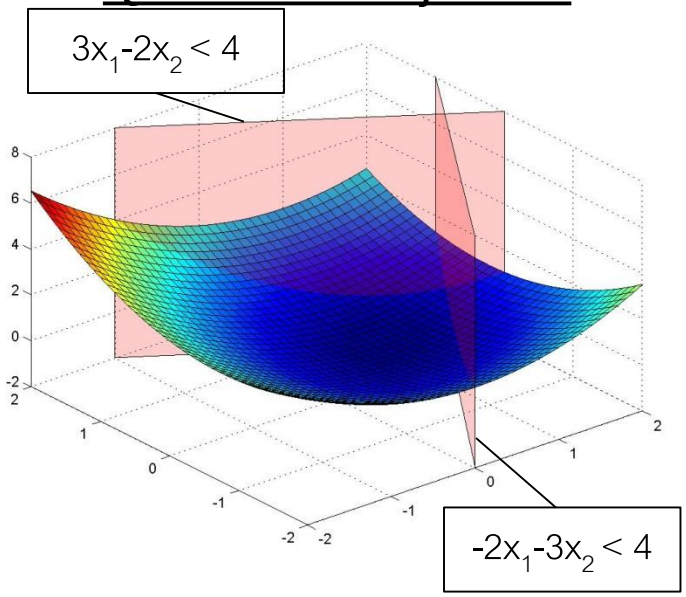
Constraint Types

- Unconstrained
- Bound
- Linear
- General Smooth
- Discrete (integer)

Linear Objective



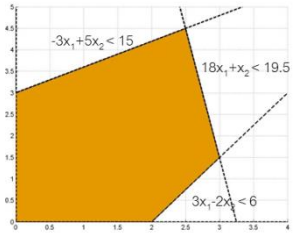
Quadratic Objective



Optimization problems

Linear

- LINPROG

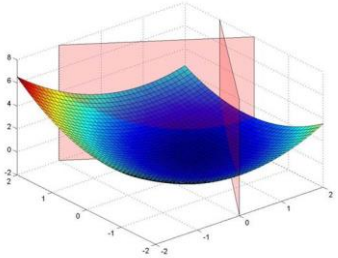


Mixed Integer

- INTLINPROG

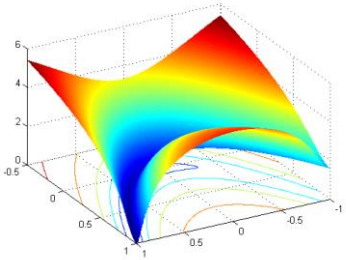
Quadratic

- QUADPROG



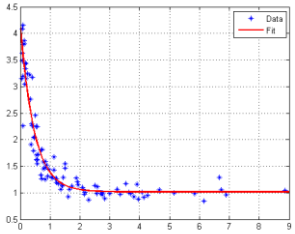
Nonlinear

- FMINCON
- MULTISTART
- GLOBALSEARCH
- PATTERNSEARCH



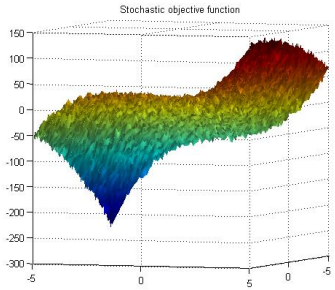
Least Squares

- LSQLIN
- LSQNONNEG
- LSQCURVEFIT
- LSQNONLIN



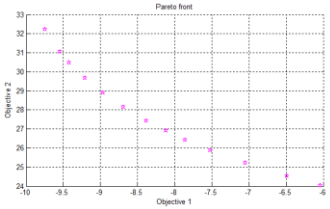
Nonsmooth or Noisy

- PATTERNSEARCH
- GA
- SIMULANNEALBND



Multiobjective

- GAMULTIOBJ



Key takeaways

1. Solve Wide Variety of Problems
   - Linear, quadratic, nonlinear, least squares (Optimization Toolbox)
   - Nonlinear, nonsmooth, stochastic, noisy (Global Optimization Toolbox)

2. Set up, Run, and Monitor Optimizations
   - Optimization App
   - MATLAB functions
   - Plot functions, command line display

3. MATLAB Environment
   - Integrated Numerics, Graphics, Symbolic Math
   - Parallel Computing