Delay-dependent $H_\infty$ control for 2-D switched delay systems in the second FM model

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Abstract

This paper is concerned with the problem of delay-dependent $H_\infty$ control for 2-D (two-dimensional) switched discrete state delay systems described by the second FM (Fornasini and Marchesini) state-space model. Firstly, some sufficient conditions for the exponential stability and weighted $H_\infty$ disturbance attenuation performance of the underlying system are derived via the average dwell time approach. Then, based on the obtained results, a state feedback controller design is proposed to guarantee that the resulting closed-loop system is exponentially stable and achieves a prescribed disturbance attenuation level $\gamma$. Finally, a numerical example is provided to verify the effectiveness of the proposed method.

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1. Introduction

2-D systems exist in many practical applications, such as circuits analysis, digital image processing, signal filtering and thermal power engineering [1–4]. Thus the analysis and synthesis of 2-D systems are interesting and challenging problems, and they have received considerable attention, for example, 2-D state-space realization theory was researched in [5,6] and [7] studied the stability and 2-D optimal control theory, [8] and [9] addressed the $H_\infty$ control problem for effectively solving the noise attenuation problem for 2-D systems.

Delay phenomena of signal transmissions are frequently encountered in engineering and biological systems [10]. They are often a source of instability and performance degradation in
many control systems. The control problem for 2-D state delay systems has received some attention. Ref. [11] presented a sufficient stability condition and a stabilization method for linear 2-D discrete state delay systems. The $H_\infty$ control and filtering problems for 2-D state delay systems were widely studied by many researchers [12–19].

On the other hand, switched systems have numerous applications in many fields, such as motor engine control, constrained robotics and networked control systems [20–22]. The stability is the basic problem for switched systems and has been widely investigated in recent years [23–25], the $H_\infty$ control and filtering problems for switched system with state delay have been studied in [26–37] and [38] considered the robust $H_\infty$ output tracking control problem under asynchronous switching. Furthermore, the switch phenomenon may also occur in practical 2-D systems, for example, the thermal processes in chemical reactors, heat exchangers and pipe furnaces with two modes, can be expressed by a 2-D switched system with time delays. Recently, there have been a few reports on 2-D discrete switched systems. Ref. [39] firstly considered 2-D switched systems with arbitrary switched sequences, and the process of switch was considered as a Markov jumping one. In addition, the stabilization problem of discrete 2-D switched systems was also studied in [40,41]. However, to the best of our knowledge, no works have directly considered the problem of $H_\infty$ control for 2-D discrete switched systems with state delay via the average dwell time approach, which motivates our present study.

In this paper, we use the average dwell time approach to solve the weighted $H_\infty$ control problem of discrete-time 2-D switched systems. The main theoretical contributions are (1) We first contribute to the development of exponential stabilization for discrete-time 2-D switched systems with state delay represented by the second FM model. (2) Employing the average dwell time method, a state feedback scheme is designed to guarantee that the corresponding closed-loop system enjoys the exponential stability with weighted $H_\infty$ disturbance attenuation level smaller than a prescribed constant $\gamma$. (3) All the developed results are expressed in terms of feasibility testing of LMIs and illustrated on a representative example.

The organization of this paper is as follows. Section 2 formulates the problem and presents some preliminary results. The analysis of exponential stability with weighted $H_\infty$ disturbance attenuation performance is given in Section 3. The weighted $H_\infty$ controller design is developed in Section 4. An example is given to illustrate the effectiveness of the approach in Section 5. Finally, the paper is concluded in Section 6.

Notations: Throughout this paper, the superscript “$T$” denotes the transpose, and the notation $X \geq Y$ ($X > Y$) means that matrix $X - Y$ is positive semi-definite (positive definite, respectively). $\| \cdot \|$ denotes the Euclidean norm. $I$ represents identity matrix with appropriate dimension. $\text{diag}(a_i)$ denotes diagonal matrix with the diagonal elements $a_i$, $i = 1, 2, \ldots, n$. $X^{-1}$ denotes the inverse of $X$. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. The set of all nonnegative integers is represented by $\mathbb{Z}_+$. A 2-D signal $s$ with $s(i,j) \in \mathbb{R}^n$, $i,j \in \mathbb{Z}_+$ is said to belong to the 2-D $l_2$–space if

$$\|s\|_2 = \sqrt{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s^T(i,j)s(i,j) < \infty}$$

where $\| \cdot \|_2$ denotes the $l_2$ norm of $s$. A 2-D signal in the $l_2$–space is an energy bounded signal.
2. Problem formulation and preliminaries

Consider discrete-time 2-D switched systems with state delays described by the following second FM state space model:

\[x(i+1,j+1) = A_{1d}^{\sigma(i,j)}x(i-1,j) + A_{2d}^{\sigma(i+1,j)}x(i+1,j-1) + B_1^{\sigma(i,j)}w(i,j) + C_1^{\sigma(i,j)}u(i,j) \tag{1a}\]

\[z(i,j) = H^{\sigma(i,j)}x(i,j) + L^{\sigma(i,j)}w(i,j) \tag{1b}\]

where \(i, j \in \mathbb{Z}_+\) are horizontal and vertical coordinates, respectively; \(x(i,j) \in \mathbb{R}^n\) is the state vector, \(w(i,j) \in \mathbb{R}^r\) is the noise input which belongs to \(L_2\{[0, \infty), [0, \infty)\}\), \(z(i,j) \in \mathbb{R}^q\) is the controlled output, \(d_1\) and \(d_2\) are unknown positive integers representing delays along horizontal direction and vertical direction, respectively; \(\sigma(i) : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{N} = 1, 2, ..., N\) is the switching signal with \(N\) denoting the number of subsystems, \(\sigma(i) = k, k \in \mathbb{N}\), denotes that the \(k\)-th subsystem is active, \(A_{1d}^k, A_{2d}^k, A_{1d}^m, A_{2d}^m, B_1^k, B_1^m, C_1^k, C_1^m, H^k, L^k\) are constant matrices with appropriate dimensions. The initial condition is defined as follows:

\[X(0) = [x(-d_1, 0), x(-d_1, 1), x(-d_1, 2), \ldots \]

\[x(-d_1 + 1, 0), x(-d_1 + 1, 1), x(-d_1 + 1, 2), \ldots \]

\[x(0, 0), x(0, 1), x(0, 2), \ldots \]

\[x(0, -d_2), x(1, -d_2), x(2, -d_2), \ldots \]

\[x(0, -d_2 + 1), x(1, -d_2 + 1), x(2, -d_2 + 1), \ldots \]

\[x(1, 0), x(2, 0), x(3, 0), \ldots \]

\[(2)\]

For 2-D switched system (1a) and (1b), assume a finite set of initial conditions, i.e., there exist positive integers \(z_1 < \infty\) and \(z_2 < \infty\), such that

\[\begin{cases}
    x(i,j) = 0, \forall j \geq z_2, i = d_1, d_1 + 1, \ldots 0 \\
    x(i,j) = 0, \forall i \geq z_1, j = d_2, d_2 + 1, \ldots 0 \\
    X(0) \in l_2, \quad \text{i.e., } ||X(0)||_2 < \infty
\end{cases} \tag{3}\]

In the paper, the switch can be assumed to occur only at each sampling points of \(i\) or \(j\). The switching sequence can be described as

\[(i_0, j_0), \sigma(i_0, j_0), (i_1, j_1), \sigma(i_1, j_1), \ldots, (i_k, j_k), \sigma(i_k, j_k)) \ldots \tag{4}\]

where \((i_k, j_k)\) denotes the \(k\)-th switching instant. It should be noted that the value of \(\sigma(i,j)\) only depends upon \(i + j\) which means that \(\sigma(i,j) = \sigma(m,n)\) for all \(i + j = m + n\) (see the references [40,41]).
In order to address the issue of the delay-dependent weighted $H_\infty$ performance, we first define vectors $s(i,j), t(i,j)\in\mathbb{R}^s$ such that

\begin{equation}
\begin{cases}
x(i + 1, j + 1) = x(i, j + 1) + s(i, j + 1) \\
x(i + 1, j + 1) = x(i + 1, j) + t(i + 1, j)
\end{cases}
\end{equation}

(5)

**Definition 1.** System (1a) and 1b with $w(i,j) = 0$ is said to be exponentially stable under the switching signal $\sigma(\cdot)$, for given $z \geq 0$ and positive constant $\xi$, if there exists a positive constant $c$, such that

\[ \sum_{i+j=D} \|x(i,j)\|^2 \leq \xi e^{-c(D-z)} \sum_{i+j=z} \|x(i,j)\|^2 \]

(6)

holds for all $D > z$ and

\[ \sum_{i+j=z} \|x(i,j)\|^2 \leq \sup_{-d_1 \leq \theta_h \leq 0, i+j=z} \sum_{-d_2 \leq \theta_i \leq 0} \{ \|x(i-\theta_h,j)\|^2, \|x(i,j-\theta_i)\|^2, \|s(i-\theta_h,j)\|^2, \|t(i,j-\theta_i)\|^2 \} \]

Remark 1. From **Definition 1**, it is easy to see that when $z$ is given, $\sum_{i+j=z} \|x(i,j)\|^2_C$ will be bounded, and $\sum_{i+j=D} \|x(i,j)\|^2$ will tend to be zero exponentially as $D$ goes to infinity, which also means $\|x(i,j)\|$ will tend to be zero exponentially.

**Definition 2.** For a given scalar $\alpha > 0$, system (1a) and (1b) is said to have a weighted $H_\infty$ disturbance attenuation $\gamma$ under switching signal $\sigma(\cdot)$ if it satisfies the following conditions:

1. When $w(i,j) = 0$, system (1a) and (1b) is asymptotically stable or exponentially stable;
2. Under the zero boundary condition, it holds that

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\alpha^{i+j} \|z(i,j)\|^2_2) < \gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|w(i,j)\|^2_2, \quad \forall 0 \neq w \in l^2_2 \{ [0, \infty), [0, \infty) \} \]

(7)

where the $l_2$-norm of 2-D discrete signal $z(i,j)$ and $w(i,j)$ are defined as

\[ \|z\|^2_2 = \|z(i + 1, j)\|^2_2 + \|z(i, j + 1)\|^2_2 \]

\[ \|w\|^2_2 = \|w(i + 1, j)\|^2_2 + \|w(i, j + 1)\|^2_2 \]

(8)

**Definition 3.** For any $i + j = D \geq z = i_z + j_z$, let $N_\sigma(z, D)$ denote the switching number of $\sigma(\cdot)$ on an interval $[z, D)$. If

\[ N_\sigma(z, D) \leq N_0 + \frac{D-z}{\tau_a} \]

(9)

holds for given $N_0 \geq 0$ and $\tau_a \geq 0$, then the constant $\tau_a$ is called the average dwell time and $N_0$ is the chatter bound.
Lemma 1. For any matrices $M_1, M_2 \in \mathbb{R}^{n \times n}$ and $0 < X \in \mathbb{R}^{n \times n}$, and any integer $d \geq 0$, the following inequality holds

$$- \sum_{i = -d}^{1} s^T(i + 1, j + 1) X s(i + 1, j + 1)$$

$$\leq \xi^T(i, j + 1) \begin{bmatrix} M^T_1 + M_1 & -M^T_1 + M_2 \\ \ast & -M^T_2 - M_2 \end{bmatrix} \xi(i, j + 1) + d^T\xi(i, j + 1)Y^T X^{-1} Y \xi(i, j + 1)$$

where

$$\xi(i, j + 1) = \begin{bmatrix} x(i, j + 1) \\ x(i - d, j + 1) \end{bmatrix}, \quad Y = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$$

Lemma 2. Consider system (1) $w(i, j) = 0$, $u(i, j) = 0$ and initial condition (3). Given a positive constant $\alpha < 1$ and integers $d_1^* > 0$, $d_2^* > 0$, if there exist a set of matrices $M_{11}^k, M_{12}^k, M_{21}^k, M_{22}^k \in \mathbb{R}^{n \times n}$ and symmetric positive definite matrices $P_k, S_1^k, S_2^k \in \mathbb{R}^{n \times n}$, with $k \in \mathbb{N}$, such that

$$\begin{bmatrix} I_1^k & \alpha d_i (-M_{11}^k + M_{12}^k) & 0 & 0 & M_{11}^k & 0 \\ \ast & \alpha d_i (-M_{12}^k - M_{12}^k) & 0 & 0 & M_{12}^k & 0 \\ \ast & \ast & \Gamma_2^k & \alpha d_i (-M_{21}^k + M_{22}^k) & 0 & M_{21}^k \\ \ast & \ast & \ast & \alpha d_i (-M_{22}^k - M_{22}^k) & 0 & M_{22}^k \\ \ast & \ast & \ast & \ast & (d^*_1)^{-1} \alpha d_i S_1^k & 0 \\ \ast & \ast & \ast & \ast & \ast & (d^*_2)^{-1} \alpha d_i S_2^k \end{bmatrix}$$

$$+ \begin{bmatrix} A^k_{1d} \\ A^k_{1d} \\ A^k_{2d} \\ A^k_{2d} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} P^k + \begin{bmatrix} A^k_{1d} \\ A^k_{1d} \\ A^k_{2d} \\ A^k_{2d} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} A_{1d}^k & A_{1d}^k & A_{1d}^k & A_{1d}^k & 0 & 0 \\ A_{2d}^k & A_{2d}^k & A_{2d}^k & A_{2d}^k & 0 & 0 \end{bmatrix} < 0$$

where

$$\Gamma_1^k = -\alpha P^k_1 + \alpha d_i (M_{11}^k + M_{12}^k)$$

$$\Gamma_2^k = -\alpha (P^k - P^k_1) + \alpha d_i (M_{21}^k + M_{22}^k)$$

then, the system is exponentially stable for any switching signal with the average dwell time satisfying

$$\tau_0 > \tau_0^* = \frac{\ln \mu}{-\ln \alpha}$$

where $\mu \geq 1$ satisfies
\( P_k \leq \mu P^1, \ P^1_k \leq \mu P^1, \ S^k_1 \leq \mu S^1_1, \ S^k_2 \leq \mu S^1_2, \ \forall k,l \in \mathbb{N} \) \hspace{1cm} (12)

**Proof.** Define the following Lyapunov–Krasovskii functional for system (1a) and (1b).

\[
V_{h}^{\sigma(i,j)}(i,j) = V_{h}^{\sigma(i,j)}(i,j) + V_{v}^{\sigma(i,j)}(i,j)
\]

where

\[
V_{h}^{\sigma(i,j)}(i,j) = x^T(i,j)P_{1}^{\sigma(i,j)}x(i,j) + \sum_{\theta = -d_{1}+1}^{0} \sum_{i+\theta}^{i} s^T(r,j+1)\alpha^{i-r}S_{1}^{h}(r,j+1)
\]

\[
V_{v}^{\sigma(i,j)}(i,j) = x^T(i,j)(P_{1}^{\sigma(i,j)}-P_{1}^{\sigma(i,j)})x(i,j) + \sum_{\theta = -d_{2}+1}^{0} \sum_{j+\theta}^{j} t^T(i,r)\alpha^{j-r}S_{2}^{k}(i,r)
\]

and \( P_k, P^1_k, S^k_1, S^k_2 \) are \( n \times n \) positive definite matrices for any \( k \in \mathbb{N} \). It is clear that \( V^k(i,j) > 0 \) for \( k \in \mathbb{N} \).

\( \Pi \) is assumed that the \( k \)-th subsystem is active on the interval \([m_k, m_{k+1})\), and the \( l \)-th subsystem is active on the interval \([m_{k-1}, m_k)\). Now we consider the Lyapunov function candidate for the \( k \)-th subsystem. We have

\[
V_{h}^{k}(i+1,j+1) - \alpha V_{h}^{k}(i,j+1)
\]

\[
= x^T(i+1,j+1)P_{1}^{k}x(i+1,j+1) - \alpha x^T(i,j+1)P_{1}^{k}x(i,j+1)
\]

\[
+ \sum_{\theta = -d_{1}+1}^{0} \sum_{i+\theta}^{i} s^T(r,j+1)\alpha^{i-r}S_{1}^{k}(r,j+1)
\]

\[
- \alpha \sum_{\theta = -d_{1}+1}^{0} \sum_{i+\theta}^{i} s^T(r,j+1)\alpha^{i-r}S_{1}^{k}(r,j+1)
\]

\[
\leq x^T(i+1,j+1)P_{1}^{k}x(i+1,j+1) - x^T(i,j+1)\alpha P_{1}^{k}x(i,j+1)
\]

\[
+ d_{1}s^T(i,j+1)S_{1}^{k}s(i,j+1) - \alpha d_{1} \sum_{r = i-d_{1}}^{i-1} s^T(r,j+1)S_{1}^{k}s(r,j+1)
\]

\[
V_{v}^{k}(i+1,j+1) - \alpha V_{v}^{k}(i,j+1)
\]

\[
= x^T(i+1,j+1)(P_{1}^{k}-P_{1})x(i+1,j+1) - \alpha x^T(i,j+1)(P_{1}^{k}-P_{1})x(i+1,j)
\]

\[
+ \sum_{\theta = -d_{2}+1}^{0} \sum_{j+\theta}^{j} t^T(i+1,r)\alpha^{j-r}S_{2}^{k}t(i+1,r)
\]

\[
- \alpha \sum_{\theta = -d_{2}+1}^{0} \sum_{j+\theta}^{j} t^T(i+1,r)\alpha^{j-r}S_{2}^{k}t(i+1,r)
\]

\[
\leq x^T(i+1,j+1)(P_{1}^{k}-P_{1})x(i+1,j+1) - x^T(i,j+1)\alpha(P_{1}^{k}-P_{1})x(i+1,j)
\]

\[
+ d_{2}t^T(i+1,j)S_{2}^{k}t(i+1,j) - \alpha d_{2} \sum_{r = j-d_{2}}^{j-1} t^T(i+1,r)S_{2}^{k}t(i+1,r)
\]

\[
V_{h}^{k}(i+1,j+1) + V_{v}^{k}(i+1,j+1) - \alpha(V_{h}^{k}(i,j+1) + V_{v}^{k}(i+1,j))
\]
Applying Lemma 1, we have the following inequalities:

\[
\begin{align*}
&\sum_{r = i-d_1}^{i-1} s^T(r, j+1)S_1^ks(r, j+1) - \alpha d_1 \sum_{r = j-d_2}^{j-1} t^T(i+1, r)S_2^kt(i+1, r) \\
&\leq \sum_{r = i-d_1}^{i-1} s^T(r, j+1)S_1^ks(r, j+1) - \alpha d_1 \sum_{r = j-d_2}^{j-1} t^T(i+1, r)S_2^kt(i+1, r)
\end{align*}
\]

Using Eq. (5), we have

\[
\begin{align*}
&\begin{bmatrix} s(i, j+1) \\ t(i+1, j) \end{bmatrix}^T \begin{bmatrix} d_1S_1^k \\ 0 \\ d_2S_2^k \end{bmatrix} \begin{bmatrix} s(i, j+1) \\ t(i+1, j) \end{bmatrix} \\
&= \begin{bmatrix} x(i, j+1) \\ x(i-d_1, j+1) \\ x(i+1, j) \\ x(i+1, j-d_2) \end{bmatrix}^T \begin{bmatrix} A_1^{kT} - I & A_1^{kT} \\ A_{1d}^{kT} & A_{1d}^{kT} \\ A_2^{kT} & A_2^{kT} - I \\ A_{2d}^{kT} & A_{2d}^{kT} \end{bmatrix} \begin{bmatrix} d_1S_1 \\ 0 \\ d_2S_2 \end{bmatrix}
\end{align*}
\]

Using Lemma 1, we have the following inequalities:

\[
\begin{align*}
&- \sum_{r = i-d_1}^{i-1} s^T(r, j+1)S_1^ks(r, j+1) \\
&\leq \sum_{r = i-d_1}^{i-1} s^T(r, j+1)S_1^ks(r, j+1) - \alpha d_1 \sum_{r = j-d_2}^{j-1} t^T(i+1, r)S_2^kt(i+1, r)
\end{align*}
\]

and

\[
\begin{align*}
&- \sum_{r = j-d_2}^{j-1} t^T(i+1, r)S_2^kt(i+1, r)
\end{align*}
\]
where $M_{11}^k, M_{12}^k, M_{21}^k, M_{22}^k \in \mathbb{R}^{n \times n}$ with $k \in \mathbb{N}$.

Substituting Eqs. (17–19) into Eq. (16) yields

$$
V_h^k(i + 1, j + 1) + V_v^k(i + 1, j + 1) - \alpha(V_h^k(i, j + 1) + V_v^k(i, j + 1))
$$

$$
= \begin{bmatrix}
    x(i, j + 1) \\
    x(i - d_1, j + 1) \\
    x(i + 1, j) \\
    x(i + 1, j - d_2)
\end{bmatrix}^T
\begin{bmatrix}
    A_1^k \\
    A_1^T \\
    A_2^k \\
    A_2^T
\end{bmatrix}
\begin{bmatrix}
    A_1^k \\
    A_1^T \\
    A_2^k \\
    A_2^T
\end{bmatrix}
\begin{bmatrix}
    x(i, j + 1) \\
    x(i - d_1, j + 1) \\
    x(i + 1, j) \\
    x(i + 1, j - d_2)
\end{bmatrix}
$$

$$
+ d_1 S_1
\begin{bmatrix}
    A_1^k & I \\
    A_1^T & A_1^k
\end{bmatrix}^T
\begin{bmatrix}
    A_1^k \\
    A_1^T \\
    A_2^k \\
    A_2^T
\end{bmatrix}
\begin{bmatrix}
    A_1^k & I \\
    A_1^T & A_1^k
\end{bmatrix}
\begin{bmatrix}
    x(i, j + 1) \\
    x(i - d_1, j + 1) \\
    x(i + 1, j) \\
    x(i + 1, j - d_2)
\end{bmatrix}
$$

where

$$
\Pi =
\begin{bmatrix}
    -\alpha P_k + \alpha^d (M_{11}^T + M_{11}^k) & \alpha^d (-M_{11}^T + M_{12}^k) \\
    \alpha^d (-M_{11}^T + M_{11}^k) & \alpha^d (-M_{12}^T - M_{12}^k)
\end{bmatrix}
\begin{bmatrix}
    -\alpha P_k - P_k^T + \alpha^d (M_{21}^T + M_{21}^k) & \alpha^d (-M_{21}^T + M_{22}^k) \\
    \alpha^d (-M_{21}^T + M_{21}^k) & \alpha^d (-M_{22}^T - M_{22}^k)
\end{bmatrix}
$$

Applying Schur complement, it follows from the LMI (10) that

$$
V_h^k(i + 1, j + 1) - \alpha V_h^k(i, j + 1) + V_v^k(i + 1, j + 1) - \alpha V_v^k(i, j + 1) < 0
$$

(21)
Thus, it is easy to get that
\[ V_h^k(i+1,j+1) + V_h^k(i+1,j+1) < \alpha (V_h^k(i,j+1) + V_h^k(i+1,j)) \] (22)

Since for any nonnegative integer \( D > z = \max(z_1, z_2) \), we have \( V_h^k(0,D) = V_h^k(D,0) = 0 \), one can get
\[
\sum_{i+j = D} V_h^k(i,j) \\
= V_h^k(0,D) + V_h^k(1,D-1) + V_h^k(2,D-2) + \cdots + V_h^k(D-1,1) + V_h^k(D,0) \\
+ V_v^k(0,D) + V_v^k(1,D-1) + V_v^k(2,D-2) + \cdots + V_v^k(D-1,1) + V_v^k(D,0) \\
< \alpha \left\{ 0 + V_h^k(0,D-1) + V_h^k(1,D-2) + \cdots + V_h^k(D-2,1) + V_h^k(D-1,0) \\
+ V_v^k(0,D-1) + V_v^k(1,D-2) + V_v^k(2,D-3) + \cdots + V_v^k(D-1,0) + 0 \right\} \\
= \alpha \sum_{i+j = D-1} V_h^k(i,j) < \cdots < \alpha^{D-z} \sum_{i+j = z} V_h^k(i,j) \] (23)

Now let \( \nu = N_\sigma(z, D) \) denote the switching number of \( \sigma(\cdot) \) on an interval \((z, D)\), and let \((i_{k-\nu+1}, j_{k-\nu+1}), (i_{k-\nu+2}, j_{k-\nu+2}), \ldots, (i_{k-1}, j_{k-1}), (i_k, j_k)\) denote the switching points of \( \sigma(\cdot) \) over the interval \((z, D)\), denoting \( m_p = l_p + j_p \), \( p = \kappa - \nu + 1, \kappa - \nu + 2, \ldots, \kappa \), thus, for \( D \in [m_k, m_{k+1}] \), it holds from Eq. (14) that
\[
\sum_{i+j = D} V^{\sigma(i,j)}(i,j) < \alpha^{D-m_k} \sum_{i+j = m_k} V^{\sigma(i,j)}(i,j) \] (24)

Using Eqs. (12) and (13), at switching instant \( m_k = i + j \), we have
\[
\sum_{i+j = m_k} V^{\sigma(i,j)}(i,j) \leq \mu \sum_{i+j = m_k} V^{\sigma(i-1,j-k-1)}(i,j) \] (25)

In addition, according to Definition 3, it follows that
\[
\nu = N_\sigma(z, D) \leq N_0 + \frac{D-z}{\tau_a} \] (26)

Therefore, the following inequality can be obtained easily.
\[
\sum_{i+j = D} V^{\sigma(i,j)}(i,j) < \alpha^{D-m_k} \sum_{i+j = m_k} V^{\sigma(i,j)}(i,j) \\
\leq \mu \alpha^{D-m_k} \sum_{i+j = m_k} V^{\sigma(i-1,j-k-1)}(i,j) \]
<\mu\alpha^{D-m_{i-1}} \sum_{i+j = m_{i-1}} V^{\sigma(i-1,i-1)}(i,j)\alpha^{m_{i-2}} - Z > \\
= \mu\alpha^{D-m_{i-1}} \sum_{i+j = m_{i-1}} V^{\sigma(i-1,i-1)}(i,j) \\
\leq ... \\
<\mu^{n-1}\alpha^{D-m_{i+1}} \sum_{i+j = m_{i+1}} V^{\sigma(i+1,i+1)}(i,j) \\
\leq \mu\alpha^{D-m_{i+1}} \sum_{i+j = m_{i+1}} V^{\sigma(i+1,i+1)}(i,j) \\
= \mu\alpha^{D-m_{i+1}} \sum_{i+j = z} V^{\sigma(i+1,i+1)}(i,j)\alpha^{m_{i+2}} - Z \\
\leq \mu\alpha^{D-z} \sum_{i+j = z} V^{\sigma(i+1,i+1)}(i,j) \\
= \mu^{n}\alpha^{D-}\sum_{i+j = z} V^{\sigma(i+1,i+1)}(i,j) \\
(27) \\

Eq. (27) can be rewritten as follows:
\\n\sum_{i+j = D} V^{\sigma(i,i)}(i,j) \leq e^{-(\frac{\lambda}{\mu} - \ln \alpha)(D-z)} \sum_{i+j = z} V^{\sigma(i+1,i+1)}(i,j) \\
(28) \\

Moreover, by Eq. (13), we can find two positive scalars \(\lambda_1, \lambda_2\) such that the following equation holds.
\\n\lambda_1 \|x(i,j)\|^2 \leq V(x(i,j)) \leq \lambda_2 \|x(i,j)\|^2 \\
(29) \\

where
\\n\lambda_1 = \min_{k \in N}(\lambda_{\min}(P^k) + \lambda_{\min}(P^k - P^k)) \\
\lambda_2 = \max_{k \in N}(\lambda_{\max}(P^k) + \lambda_{\max}(P^k - P^k) + 2\alpha(d_1 + 1) + 2\alpha(d_2 + 1)\lambda_{\max}(S^k)) \\

Then from Eqs. (28) and (29), we have
\\n\sum_{i+j = D} \|x(i,j)\|^2 \leq \frac{\lambda_2}{\lambda_1} e^{-(\frac{\lambda}{\mu} - \ln \alpha)(D-z)} \sum_{i+j = z} \|x(i,j)\|^2 \\
(30) \\

By Definition 1, we know that if \(-\ln(\mu/\tau_a) - \ln \alpha > 0\), that is \(\tau_a > \tau_a^* = \ln \mu/(-\ln \alpha)\), the 2-D discrete switched system is exponentially stable.

The proof is completed.

Remark 2. Note that when \(\mu = 1\) in Eq. (11), then (12) is reduced to be, \(P^k = P^k, P^k = P^k\), \(S^k = S^k, S^k = S^k, \forall k, l \in N\). In the case, we have \(\tau_a \geq \tau_a^* = 0\), which means that the switching signal can be arbitrary.

3. Stability and \(H_\infty\) performance analysis.

Theorem 1. Consider system (1) with \(w(i,j) = 0, u(i,j) = 0\) and initial condition (3). Given a positive constant \(\alpha < 1\) and integers \(d_1^k > 0, d_2^k > 0\), if there exist a set of matrices \(M^{k}_{11}, M^{k}_{12}, M^{k}_{21}\).
\( M_{22}^k \in \mathbb{R}^{n \times n} \) and symmetric positive definite matrices \( P_k, P^k_1, S^k_1, S^k_2 \in \mathbb{R}^{n \times n} \) with \( k \in \mathbb{N} \) such that

\[
\begin{bmatrix}
A_{11}^{kT} & A_{12}^{kT} \\
A_{21}^{kT} & A_{22}^{kT}
\end{bmatrix} + \begin{bmatrix}
A_{11}^{kT} & A_{12}^{kT} \\
A_{21}^{kT} & A_{22}^{kT}
\end{bmatrix}^T \begin{bmatrix}
A_{11}^{kT} - I & A_{12}^{kT} \\
A_{21}^{kT} & A_{22}^{kT} - I
\end{bmatrix} \begin{bmatrix}
\alpha \\ \alpha
\end{bmatrix} = \begin{bmatrix}
\alpha \\ \alpha
\end{bmatrix}^T \begin{bmatrix}
\alpha \\ \alpha
\end{bmatrix}
\]

where

\[
\Xi_1^k = H^{kT} H^k - \alpha P_k^k + \alpha^2 (M_{11}^{kT} + M_{11}^k)
\]

\[
\Xi_2^k = H^{kT} H^k - \alpha (P_k^k - P_1^k) + \alpha^2 (M_{21}^{kT} + M_{21}^k)
\]

\[
W_1^k = \alpha^2 (-M_{11}^{kT} + M_{12}^k), \quad W_2^k = \alpha^2 (-M_{12}^{kT} - M_{12}^k)
\]

\[
W_3^k = \alpha^2 (-M_{21}^{kT} + M_{22}^k), \quad W_4^k = \alpha^2 (-M_{22}^{kT} - M_{22}^k)
\]

then the corresponding 2-D switched system has an weighted \( H_\infty \) noise attenuation level \( \gamma \) for any switching signal with the average dwell time satisfying

\[
\tau_a > \tau^*_a = \frac{\ln \mu}{-\ln \alpha}
\]

where \( \mu \geq 1 \) satisfies

\[
P_k^l \leq \mu P_l^k, \quad P_1^k \leq \mu P_1^k, \quad S_1^k \leq \mu S_1^k, \quad S_2^k \leq \mu S_2^k, \quad \forall k, l \in \mathbb{N}
\]

**Proof.** To establish the weighted \( H_\infty \) performance of system (1a) and (1b) with the control input \( u(i, j) = 0 \) for \( w(i, j) \in \mathbb{L}_2 \{[0, \infty], [0, \infty]\} \), we consider

\[
V_h^k(i + 1, j + 1) + V_h^k(i + 1, j + 1) - \alpha (V_h^k(i, j + 1) + V_h^k(i + 1, j))
\]

\[
+ \left[ z(i, j + 1) \right]^T \left[ z(i, j + 1) \right] - (1 - \gamma^2) \left[ w(i, j + 1) \right]^T \left[ w(i, j + 1) \right]
\]

\[
+ \left[ z(i + 1, j) \right]^T \left[ z(i + 1, j) \right] - (1 - \gamma^2) \left[ w(i + 1, j) \right]^T \left[ w(i + 1, j) \right]
\]

\[
\left[ z(i + 1, j) \right]^T \left[ z(i + 1, j) \right] - (1 - \gamma^2) \left[ w(i + 1, j) \right]^T \left[ w(i + 1, j) \right]
\]
where

\[
\Pi = \begin{bmatrix}
\Xi^k_1 & W^k_1 & 0 & 0 & H^{kT}L^k & 0 \\
* & W^k_2 & 0 & 0 & 0 & 0 \\
* & * & \Xi^k_2 & W^k_3 & 0 & H^{kT}L^k \\
* & * & * & W^k_4 & 0 & 0 \\
* & * & * & * & L^{kT}L^k-(1-\tau)y^2I & 0 \\
* & * & * & * & * & L^{kT}L^k-(1-\tau)y^2I \\
\end{bmatrix}
\]

Applying Schur complement, it follows from the LMI (31) that

\[
V^k(i+1,j+1) - \alpha V^k_h(i,j+1) - \alpha V^k_h(i+1,j) \\
+ \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix}^T \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix} - \gamma^2 \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \end{bmatrix}^T \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \end{bmatrix} < 0
\]

We take

\[
\bar{z} = \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix}, \ \bar{w} = \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \end{bmatrix}
\]
Then we have

\[ V^k(i + 1, j + 1) < \alpha (V^k_h(i, j + 1) + V^k_v(i + 1, j)) + \mathbf{z}^T \mathbf{z} - \gamma^2 \mathbf{w}^T \mathbf{w} \]  

(36)

Set

\[ \Gamma(i, j) = \| \mathbf{z} \|_2^2 - \gamma^2 \| \mathbf{w} \|_2^2 \]

Applying zero boundary condition, one gets

\[
\sum_{i+j = D} V^{\sigma(i, j, k)}(i, j) < \alpha \sum_{i+j = D-1} V^{\sigma(i, j, k)}(i, j) - \sum_{i+j = D-2} \Gamma(i, j)
\]

\[
< \alpha^{D-m_k} \sum_{i+j = m_k} V^{\sigma(i, j, k)}(i, j) \sum_{m = m_k-1}^{D-2} \sum_{i+j = m} \alpha^{D-2-m} \Gamma(i, j)
\]

\[
\leq \mu \alpha^{D-m_k} \sum_{i+j = (m_k)} V^{\sigma(i, j, k)}(i, j) \sum_{m = m_k-1}^{D-2} \sum_{i+j = m} \alpha^{D-2-m} \Gamma(i, j)
\]

\[
< \sum_{i+j = m_k-1} \mu^{N_k(i+j, D)} \alpha^{D-(m_k-1)} V^{\sigma(i, j, k-1)}(i, j)
\]

\[
- \mu \alpha^{D-m_k} \sum_{i+j = m_k-2} \Gamma(i, j) \sum_{m = m_k-1}^{D-2} \sum_{i+j = m} \alpha^{D-2-m} \Gamma(i, j)
\]

\[
= \sum_{i+j = m_k-1} \mu^{N_k(i, j, D)} \alpha^{D-(m_k-1)} V^{\sigma(i, j, k-1)}(i, j)
\]

\[
- \sum_{m = m_k-1}^{D-2} \sum_{i+j = m} \mu^{N_k(i, j+1, D)} \alpha^{D-2-m} \Gamma(i, j)
\]

\[
< \sum_{i+j = (m_k-1)} \mu^{N_k(i, j, D)} \alpha^{D-(m_k-1)} V^{\sigma(i, j-1, k-1)}(i, j)
\]

\[
- \sum_{m = m_k-1}^{D-2} \sum_{i+j = m} \mu^{N_k(i, j+1, D)} \alpha^{D-2-m} \Gamma(i, j)
\]

\[
< \sum_{i+j = 0} \mu^{N_k(i, j+1, D)} \alpha^{D-1} V^{\sigma(0,1)}(i, j) - \sum_{m = 0}^{D-2} \sum_{i+j = m} \mu^{N_k(i, j+1, D)} \alpha^{D-2-m} \Gamma(i, j)
\]  

(37)
Under the zero initial condition, it holds that
\[ \sum_{i+j=1}^{\infty} \mu^{N_a(i+j,D)} D^{-1} V^{\sigma(0,1)}(i,j) = 0 \] (38)

So we have
\[ \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{N_a(i+j+1,D)} D^{-2-i-j} \Gamma(i,j) < - \sum_{i+j=D}^{\infty} V^{\sigma(i,j)}(i,j) < 0 \] (39)

Multiplying the both sides of Eq. (39) by \( \mu^{-N_a(1,D)} \), we can get the following inequality:
\[ \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{-N_a(1,i+j+1)} D^{-2-i-j} ||z||_2^2 < \gamma^2 \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{-N_a(1,i+j+1)} D^{-2-m} ||w||_2^2 \] (40)

That is
\[ \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{-N_a(1,i+j+1)} D^{-2-m} ||z||_2^2 < \gamma^2 \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{-N_a(1,i+j+1)} D^{-2-m} ||w||_2^2 \] (41)

Noting \( N_a(1, i+j+1) \leq (i+j)/\tau_a \), and using Eq. (39), we have
\[ \mu^{-N_a(1,i+j+1)} = e^{-N_a(1,i+j+1) \ln \mu} \geq e^{(i+j) \ln \alpha} \] (42)

Thus
\[ \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} e^{(i+j) \ln \alpha} D^{-2-i-j} ||z||_2^2 < \gamma^2 \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \mu^{-N_a(1,i+j+1)} D^{-2-m} ||w||_2^2 \]

\[ \sum_{D=2}^{\infty} \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \alpha^{D-2-i-j} ||z||_2^2 < \gamma^2 \sum_{D=2}^{\infty} \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \alpha^{D-2-i-j} ||w||_2^2 \]

\[ \sum_{D=2}^{\infty} \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \alpha^{D-2-i-j} ||z||_2^2 < \gamma^2 \sum_{D=2}^{\infty} \sum_{m=0}^{D-2} \sum_{i+j=m}^{\infty} \alpha^{D-2-i-j} ||w||_2^2 \]

\[ \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} \alpha^{i+j} ||z||_2^2 < \gamma^2 \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} \alpha^{D-2-m} ||w||_2^2 \]

\[ \frac{1}{1-\alpha} \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} \alpha^{i+j} ||z||_2^2 < \gamma^2 \frac{1}{1-\alpha} \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} ||w||_2^2 \]

\[ \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} \alpha^{i+j} ||z||_2^2 < \gamma^2 \sum_{m=0}^{\infty} \sum_{i+j=m}^{\infty} ||w||_2^2 \]

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha^{i+j} ||z||_2^2 < \gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} ||w||_2^2 \] (43)

According to Definition 3, we can obtain that system (1a) and (1b) is exponentially stable and has a prescribed weighted \( H_\infty \) disturbance attenuation level \( \gamma \).

The proof is completed.
4. Delay-dependent $H_{\infty}$ controller design

Consider system (1a) and (1b) and the following controller:

$$u(i,j) = K^{\sigma(i,j)}x(i,j)$$

(44)

The corresponding closed-loop system is given by

$$x(i+1,j+1) = (A_{1}^{\sigma(i,j+1)} + C_{1}^{\sigma(i,j+1)}K^{\sigma(i,j+1)})x(i,j) + (A_{2}^{\sigma(i,j+1)} + C_{2}^{\sigma(i,j+1)}K^{\sigma(i,j+1)})x(i+1,j) + A_{1d}^{\sigma(i,j+1)}x(i-d_{1},j+1) + A_{2d}^{\sigma(i,j+1)}x(i+1,j-d_{2}) + B_{1}^{\sigma(i,j+1)}w(i,j+1) + B_{2}^{\sigma(i,j+1)}w(i+1,j)$$

(45a)

$$z(i,j) = H^{\sigma(i,j)}x(i,j) + L^{\sigma(i,j)}w(i,j)$$

(45b)

If there exists a controller as described in Eq. (44) such that the closed-loop system (45a) and (45b) is exponentially stable and has a specified weighted $H_{\infty}$ noise attenuation level $\gamma$.

**Theorem 2.** Given a positive constant $\alpha < 1$ and integers $d_{1}^{k} > 0$, $d_{2}^{k} > 0$, if there exist a set of matrices $N_{11}^{k}, N_{12}^{k}, N_{21}^{k}, N_{22}^{k}\in\mathbb{R}^{n\times n}$, $N^{k}\in\mathbb{R}^{m\times n}$ and symmetric positive definite matrices $\bar{\mathcal{P}}^{k}, \bar{\mathcal{P}}^{1}, \bar{\mathcal{S}}^{k}, \bar{\mathcal{S}}^{1}\in\mathbb{R}^{n\times n}$, with $k\in\mathbb{N}_{+}$, such that.

$$\begin{bmatrix}
\Pi^{k} & \Omega_{1}^{kT} & \Omega_{2}^{kT} & \Omega_{3}^{kT} & \Omega_{4}^{kT} & \Omega_{5}^{kT} & \Omega_{6}^{kT} \\
* & -I & 0 & 0 & 0 & 0 & 0 \\
* & * & -P^{k} & 0 & 0 & 0 & 0 \\
* & * & * & -d_{1}^{1}\bar{S}^{1} & 0 & 0 & 0 \\
* & * & * & * & -d_{1}^{2}\bar{S}^{2} & 0 & 0 \\
* & * & * & * & * & -\alpha^{-\theta}d_{1}^{1}\bar{S}^{1} & 0 \\
* & * & * & * & * & * & -\alpha^{-\theta}d_{2}^{2}\bar{S}^{2}
\end{bmatrix} < 0$$

(46)

where

$$\Pi^{k} = \begin{bmatrix}
-\alpha\bar{P}^{1} & \alpha^{\theta}(\bar{P}^{k} + N_{11}^{kT}) & 0 & 0 & 0 & 0 \\
* & -\alpha^{\theta}(N_{12}^{k} + N_{12}^{kT}) & 0 & 0 & 0 & 0 \\
* & * & -\alpha(\bar{P}^{k} - \bar{P}^{1}) & \alpha^{\theta}(\bar{P}^{k} + N_{21}^{kT}) & 0 & 0 \\
* & * & * & -\alpha\bar{d}_{1}^{\theta}(N_{22}^{k} + N_{22}^{kT}) & 0 & 0 \\
* & * & * & * & -\gamma^{2}I & 0 \\
* & * & * & * & * & -\gamma^{2}I
\end{bmatrix},$$

$$\Omega_{1}^{k} = \begin{bmatrix}
H^{k}\bar{P}^{k} & 0 & 0 & L^{k} & 0
\end{bmatrix},$$

$$\Omega_{2}^{k} = \begin{bmatrix}
0 & 0 & H^{k}\bar{P}^{k} & 0 & 0 & L^{k}
\end{bmatrix},$$

$$\Omega_{3}^{k} = \begin{bmatrix}
A_{1d}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{4}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{5}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{6}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{7}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{8}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix},$$

$$\Omega_{9}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix}. $$

$$\Omega_{10}^{k} = \begin{bmatrix}
A_{1d}^{k}\bar{P}^{k} + C_{1}^{k}N^{k} - A_{1d}^{k}\bar{N}_{11}^{k} & -\bar{P}^{k} & A_{1d}^{k}\bar{N}_{12}^{k} & A_{2d}^{k}\bar{P}^{k} + C_{2}^{k}N^{k} - A_{2d}^{k}\bar{N}_{21}^{k} & A_{2d}^{k}\bar{N}_{22}^{k} & B_{1}^{k} & B_{2}^{k}
\end{bmatrix}.$$
\[ \Omega_5^k = \begin{bmatrix} A_{1d}^k P^k + C_1^k N^k - A_{1d}^k N_{11}^k & A_{1d}^k N_{12}^k & A_{2d}^k P^k + C_2^k N^k - A_{2d}^k N_{21}^k - P^k & A_{2d}^k N_{22}^k & B_1^k & B_2^k \end{bmatrix}, \]
\[ \Omega_6^k = \begin{bmatrix} 0 & S_1^k & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ \Omega_7^k = \begin{bmatrix} 0 & 0 & 0 & S_2^k & 0 & 0 \end{bmatrix}, \]

then, the corresponding 2-D switched system (45a) and (45b) has a weighted $H_\infty$ noise attenuation level $\gamma$ for any switching signal with the average dwell time satisfying

\[ \tau_a > \tau_a^* = \frac{\ln \mu}{\ln \alpha} \]

(47)

where $\mu \geq 1$ satisfies

\[ P^k \leq \mu P^k_1, \quad S_1^k \leq \mu S_1^k, \quad S_2^k \leq \mu S_2^k, \quad \forall k \in \mathbb{N} \]

(48)

with $P^k_1 = (P^k)^{-1} P^k (P^k)^{-1}$, the controller parameters can be obtained by $K^k = N^k (P^k)^{-1}$.

**Proof.** By applying Theorem 1 and Schur complement, a sufficient condition for the closed-loop system (45a) and (45b) to have a specified weighted $H_\infty$ noise attenuation level $\gamma$ is that there exist matrices $M_{11}^k, M_{12}^k, M_{21}^k, M_{22}^k \in \mathbb{R}^{n \times n}$ and positive definite matrices $P^k, P^k_1, S_1^k, S_2^k \in \mathbb{R}^{n \times n}$ satisfying $P^k - P^k_1 > 0$, with $k \in \mathbb{N}$, such that

\[
\begin{bmatrix} X_1 & W_1 & 0 & 0 & 0 & 0 & H_{11} & 0 & A_{11}^T
A_{1d} & W_2 & 0 & 0 & 0 & 0 & 0 & H_{12} & A_{12}^T
A_{1d}^T & A_{1d} & W_3 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_4 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_5 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_6 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_7 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_8 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
A_{1d}^T & A_{1d} & W_9 & 0 & 0 & 0 & 0 & 0 & A_{12}^T
\end{bmatrix} < 0
\]

(49)
where
\[
\Xi^k = -\alpha P^k_1 + \alpha^d (M^k_{11} + M^k_{11})
\]
\[
\Xi^k_2 = -\alpha (P^k - P^k_1) + \alpha^d (M^k_{21} + M^k_{21})
\]
\[
W^k_1 = \alpha^d (-M^k_{11} + M^k_{12})
\]
\[
W^k_2 = \alpha^d (-M^k_{12} - M^k_{12})
\]
\[
W^k_3 = \alpha^d (-M^k_{2T} + M^k_{22})
\]
\[
W^k_4 = \alpha^d (-M^k_{2T} - M^k_{22})
\]
It follows from Eq. (49) that \(M^k_{12}, M^k_{22}\) are reversible. Introducing
\[
\Phi^k_1 = \begin{bmatrix}
(P^k)^{-1} & 0 \\
-(M^k_{12})^{-1}M^k_{11}(P^k)^{-1} & (M^k_{12})^{-1}
\end{bmatrix}
\]
\[
\Phi^k_2 = \begin{bmatrix}
(P^k)^{-1} & 0 \\
-(M^k_{22})^{-1}M^k_{21}(P^k)^{-1} & (M^k_{22})^{-1}
\end{bmatrix}
\]
Pre- and post-multiplying the matrix inequality (49) by the matrix
\[
diag\{\Phi^k_1, \Phi^k_2, I, I, I, I, d^k_1, d^k_2, d^k_1(S^k_1)^{-1}, d^k_2(S^k_2)^{-1}\}
\]
and its transpose, and denoting
\[
\bar{P}^k = (P^k)^{-1}, \quad \bar{P}^k_1 = (P^k)^{-1}P^k_1(P^k)^{-1}, \quad \bar{S}^k_1 = (S^k_1)^{-1}, \quad \bar{S}^k_2 = (S^k_2)^{-1},
\]
\[
N^k_{11} = (M^k_{12})^{-1}M^k_{11}(P^k)^{-1}, \quad N^k_{12} = (M^k_{12})^{-1}, \quad N^k_{21} = (M^k_{22})^{-1}M^k_{21}(P^k)^{-1},
\]
\[
N^k_{22} = (M^k_{22})^{-1}, \quad \text{and} \quad N^k = K^k\bar{P}^k,
\]
we can see that Eq. (49) is equivalent to the LMI (46). It follows from Theorem 1 that the claim of this theorem is true.

This completes the proof.

5. Illustrative example

In this section, we present an example to illustrate the effectiveness of the proposed approach. Consider the thermal processes in chemical reactors with two modes, which can be expressed in the following partial differential equation with time delays. We assume that one can switch from a mode to another arbitrarily.

\[
\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0^{\sigma(x,t)}T(x,t) - a_1^{\sigma(x,t)}T(x,t-\tau) + b^{\sigma(x,t)}u(x,t)
\]

(50)

where \(T(x,t)\) is the temperature at \(x\in[0,x_0]\) (space) and \(t\in[0,\infty]\) (time), \(u(x,t)\) is the input function, \(\tau\) is the time delay, and \(a_0^{\sigma(x,t)}, a_1^{\sigma(x,t)}, b^{\sigma(x,t)}\) are real coefficients with \(\sigma(x,t)\) denoting the working subsystem at \((x,t)\).

Taking
\[
T(i,j) = T(i\Delta x, j\Delta t), \quad u(i,j) = u(i\Delta x, j\Delta t), \quad \sigma(i,j) = \sigma(i\Delta x, j\Delta t),
\]
\[
\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j)-T(i-1,j)}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T(i+1,j)-T(i,j)}{\Delta t},
\]

(50)
we can rewrite Eq. (50) in the discrete form

\[
T(i,j + 1) = \left(1 - \frac{\Delta t}{\Delta x} - \sigma(i,j)\right) T(i,j) + \frac{\Delta t}{\Delta x} T(i - 1, j) - \sigma(i,j) \Delta t T(i,j - d_2) + b^{s(i,j)} u(x,t)
\]

It should be noted that the value of \( \sigma(i,j) \) only depends on \( i + j \) (see the Refs. [40,41]). Then the system can be converted into a 2-D FM state space model (1a) and (1b) with parameters as follows.

Subsystem 1:

\[
A_1^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.65 \end{bmatrix}, \quad A_{1d}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2d}^1 = \begin{bmatrix} 0 & 0 \\ 0 & -0.12 \end{bmatrix},
\]

\[
C_1^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_2^1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad B_1^1 = \begin{bmatrix} 0.2 \\ 0.004 \end{bmatrix}, \quad B_2^1 = \begin{bmatrix} 0.1 \\ 0.004 \end{bmatrix}, \quad H_1^1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L_1 = 0.5.
\]

Subsystem 2:

\[
A_1^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2^2 = \begin{bmatrix} 0.23 & 0 \\ 0 & 0.62 \end{bmatrix}, \quad A_{1d}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2d}^2 = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix},
\]

\[
C_1^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_2^2 = \begin{bmatrix} 0 \\ 0.12 \end{bmatrix}, \quad B_1^2 = \begin{bmatrix} 0.5 \\ 0.04 \end{bmatrix}, \quad B_2^2 = \begin{bmatrix} 0.1 \\ 0.04 \end{bmatrix}, \quad H_1^2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L_2 = 0.5.
\]

Take \( \alpha = 0.85, \gamma = 8, d_1^s = 3 \) and \( d_2^s = 4 \). According to Theorem 2, solving Eq. (46) gives rise to the following solutions:

\[
P^1 = \begin{bmatrix} 0.3658 & -0.0341 \\ -0.0341 & 0.0479 \end{bmatrix}, \quad P_1^i = \begin{bmatrix} 0.1837 & -0.0114 \\ -0.0114 & 0.0143 \end{bmatrix},
\]

\[
S_1^1 = \begin{bmatrix} 24255 & -9.000 \\ -9.000 & 24241 \end{bmatrix}, \quad S_1^i = \begin{bmatrix} 17642 & 0.000 \\ 0.000 & 0.000 \end{bmatrix},
\]

\[
N_{11}^1 = \begin{bmatrix} -0.3658 & 0.0341 \\ 0.0341 & -0.0479 \end{bmatrix}, \quad N_{12}^1 = \begin{bmatrix} 107010 & -10.000 \\ -10.000 & 10698 \end{bmatrix},
\]

\[
N_{21}^1 = \begin{bmatrix} -0.3658 & 0.0341 \\ 0.0347 & -0.0487 \end{bmatrix}, \quad N_{22}^1 = \begin{bmatrix} -118370 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},
\]

\[
N^1 = \begin{bmatrix} -0.8952 & 0.0561 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 0.3625 & -0.0265 \\ -0.0265 & 0.0380 \end{bmatrix}, \quad P_1^i = \begin{bmatrix} 0.1827 & -0.0091 \\ -0.0091 & 0.0118 \end{bmatrix},
\]

\[
S_1^2 = \begin{bmatrix} 24252 & -12.000 \\ -12.000 & 24237 \end{bmatrix}, \quad S_2^i = \begin{bmatrix} 17641 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},
\]

\[
N_{11}^2 = \begin{bmatrix} -0.3625 & 0.0265 \\ 0.0265 & -0.0380 \end{bmatrix}, \quad N_{12}^2 = \begin{bmatrix} 107000 & -20.000 \\ -20.000 & 106980 \end{bmatrix},
\]

\[
N_{21}^2 = \begin{bmatrix} -0.3625 & 0.0265 \\ 0.0269 & -0.0386 \end{bmatrix}, \quad N_{22}^2 = \begin{bmatrix} 118370 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix},
\]

\[
N^2 = \begin{bmatrix} -0.8217 & 0.0417 \end{bmatrix}.
\]
The positive scalar $\mu = 1.3129$ can be obtained by solving Eq. (48), and then $\tau^* = \ln \mu / (-\ln \alpha) = 1.6753$ can be obtained from Eq. (47). And the controller parameters can be obtained from solving $K^k = N^k(P^k)^{-1}$.

$$K^1 = \begin{bmatrix} -2.5041 & -0.6131 \end{bmatrix}, \quad K^2 = \begin{bmatrix} -2.3038 & -0.5051 \end{bmatrix}$$

Choosing $\tau_a = 2$, simulation results are shown in Figs. 1–3, where the boundary condition of the system is

$$x(i,j) = \frac{1}{5(i+1)}, \quad \forall 0 \leq j \leq 200, i = -2, -1, 0,$$

$$x(i,j) = \frac{1}{5(j+1)}, \quad \forall 0 \leq i \leq 200, j = -2, -1, 0$$
and \( w(i,j) = 5 \exp(-0.025\pi(i + j)) \). It can be seen from Figs. 1–3 that the system is exponentially stable. Furthermore, when the boundary condition is zero, by computing, we get
\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha^{i+j} \|\pi\|_2^2 = 41.9160 \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|w\|_2^2 = 697.5124, \]
and the condition (2) in Definition 2 is guaranteed. It can be observed that the system has a weighted \( H_\infty \) disturbance attenuation level \( \gamma = 8 \).

6. Conclusions

In this paper, we have studied the \( H_\infty \) state feedback control problem for discrete-time 2-D switched state-delayed systems. A sufficient condition for the exponential stability with weighted \( H_\infty \) disturbance attenuation performance has been obtained, and a 2-D switched state feedback controller design approach was developed. A numerical example was given to demonstrate the applicability of the proposed approach. Our future work will be devoted to the dynamic output feedback controller design.

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References


